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The Teaching of Secondary Mathematics

BY

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AND

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SECOND EDITION

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THE TEACHING OF SECONDARY MATHEMATICS

PREFACE TO THE SECOND EDITION

The first edition of this book was published shortly before the entry of the United States into the Second World War. The stresses of the war years wrought some profound changes in the whole pattern of secondary education, and in no area were the stresses more evident than in mathematics. Furthermore, the ending of the war did not signalize the settlement of the problems of mathematical education. Issues are being debated more widely, and perhaps more strongly, now than ever before.

The period composed of the war years and the immediate postwar years has been characterized by the publication of a vast amount of material concerning various aspects of mathematical education in the United States. The generous reception which the original edition of this book enjoyed implies an obligation on the part of the authors to keep it abreast of the times. Enough time has now elapsed to permit the issues and developments since 1940 to be viewed with the benefit of some perspective, so we feel that a revision of certain parts of the book should be made.

The form and framework of the book are kept intact, but substantial revision of some chapters has been made. This is particularly true of Chapters II, IV, and VI, which have been largely rewritten to include developments in the last ten years. Attention is given to the increasing emphasis upon general education and upon the use of multisensory aids in teaching. The lists of exercises at the ends of the chapters have been revised, and the chapter bibliographies have been thoroughly overhauled and brought up to date. We hope that these revisions will retain whatever good features the original edition may have possessed and will, at the same time, increase the usefulness of the book through the attention given to recent developments.

For their kind permission to use quotations from published works in this revision of the book, we wish to express our appreciation to the following additional publishers, organizations, and individuals: Harper & Brothers; Harvard University Press; the Michigan Section

of the Mathematical Association of America; William Betz; and Howard F. Fehr. We are also deeply indebted to Carolyn Whitaker and Melvin LeRoy Hoff for their assistance in preparing the revised manuscript

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PREFACE TO THE FIRST EDITION

Those persons who concern themselves with the teaching of mathematics in the secondary schools usually find their interests springing mainly from one or another of three sources. Students of education in general, and in particular those who are charged with responsibility for the study and appraisal of the curriculum, must give thought to the question of the place that mathematics has occupied, does occupy, and should occupy in the secondary-school program. Those who are responsible for the maintenance of effective instruction and for its improvement, for example, the supervising principal and the departmental supervisor, will be interested in the study of the characteristic instructional problems which mathematics presents, as well as in the mathematics curriculum. Finally, those whose task it is to direct or carry on the actual classroom instruction will be concerned not only with the foregoing general considerations, but also with the many special instructional problems arising from the subject matter itself, and with the specific difficulties which students often encounter in their study of mathematics. This book has been written with the hope that it may contribute something of value to each of these groups. Their problems and interests are by no means mutually exclusive, and it is hoped that, regardless of the group to which the reader may belong, he will find the entire book worthy of his consideration. For the convenience of the reader, however, the discussion has been organized in three main divisions which bear with respective emphasis upon the predominant interests indicated above.

In Part I, The Place and Function of Mathematics in Secondary Education, the discussion is designed to be of special interest to the student of education in his desire to arrive at a proper orientation of mathematics in the secondary-school curriculum. It should also serve as a significant informational and philosophical background for the administrator, supervisor, and classroom teacher.

The discussion in Part II, The Improvement and Evaluation of Instruction in Secondary Mathematics, is directed toward those problems which concern the administrator and supervisor in their efforts to improve instruction in secondary mathematics. The student of education who wishes to get a clear perspective of curricular

and instructional problems as related to mathematics on the secondary level should be concerned with the questions raised and discussed here. Certainly the classroom teacher should know something about the administrative aspects of the improvement and evaluation of instruction.

Problems of instruction in arithmetic, algebra, geometry, trigonometry, and calculus are discussed in Part III, The Teaching of the Special Subject Matter of Secondary Mathematics. The effort is made to consider aspects of the subject matter which often present major instructional problems. In the discussion the authors have drawn liberally, but with discrimination, upon the literature dealing with the subject, as well as upon their own firsthand classroom experience.

Secondary mathematics is defined to include that mathematics which is taught in the junior high school, the senior high school, and the junior college. This is in accordance with the definition given by the Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics in the report on *The Place of Mathematics in Secondary Education.* Extensive reference to this report has been made throughout the entire book. Similarly, repeated reference has been made to the Report of the Committee of the Progressive Education Association on the Function of Mathematics in General Education. The title of this latter report is *Mathematics in General Education.* The contrasting points of view of these two important recent reports have been presented and discussed.

Not infrequently writers on educational subjects tend to become unduly impressed with radical innovations and points of view. Subject matter specialists, on the other hand, often cling with equal pertinacity to traditional practice and tend to resent any innovations. The authors of this book have endeavored to avoid either of these extreme positions and to maintain a sane balance between the more significant implications of both points of view. They recognize that tradition *per se* is not reprehensible, and that one of the principal functions of education is the conservation of established values. At the same time they are keenly aware that values need not only to be conserved but also to be extended, and that instructional methods need to be improved if mathematical instruction is to be brought to its potential fruitfulness.

This book is presented in the hope that it may help to point the way toward better instruction in mathematics. The material has

been used in the classroom in manuscript form, and the authors are indebted to their students who have discussed with them many of the problems and issues raised.

The authors further wish to make grateful acknowledgment of their indebtedness to their colleagues who by helpful criticism and encouragement have assisted in the preparation of this book; and to certain publishers, organizations, periodicals, and individual authors for their courtesy and generous cooperation in granting permission to use references and quotations from copyrighted material. Special thanks are due to Miss Ferrel Locke, M. Douglas Brown, C. B. Collier, Jr., and H. L. Cook for their assistance in preparing the manuscript.

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The following periodicals were especially helpful: *Childhood Education; Educational Administration and Supervision; The Journal of Educational Psychology; The Kadelpian Review; The Mathematics Teacher; The North Central Association Quarterly; The Peabody Journal; The Peabody Reflector and Alumni News; The Review of Educational Research; The School Review; School Science and Mathematics; Teachers College Record.*

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EDITOR'S INTRODUCTION

Educational programs are always set up originally for the teaching of human beings. They are designed to give individual learners those patterns of conduct and personality which the community considers desirable. They are directed towards total behavior-changing goals. It is only after they have developed a system of organized procedures, a set of customary routines, that it becomes possible for them to impart facts for the sake of the facts themselves. Thus, by being so busy with the tools of facts that little time is left for the goals of changed human beings, the original education system sometimes becomes merely a system of pedagogical red tape.

Of course this change has to be made by teachers. They are the real operators of the system. They are the assemblers and organizers of facts and skills. They like to make logical groupings of these facts and skills for instructional purposes. They love systematic learning. They use the logical arrangement of facts and the love of learning as instruments to achieve the behavior-changing goals of the schools. They keep changing their procedures, reorganizing their facts, and modifying their methods until they get good results, or at least believe they get good results.

This is always a crucial point in the professional development of a teacher. He is standing on the spot where he can most easily forget the goal of his teaching and begin to use the tools and procedures of subject matter as ends in themselves. Nevertheless, he must know his instruments, he must have a comprehensive and exact command of subject matter, and he must have the strong interest in procedures which always goes with a high level of technical skill. Only so can he use instruments and procedures effectively in attaining desired changes in learners. Thus he must forever steer a dangerous course between the Scylla of subject-matter worship and the Charybdis of dreamy-eyed enthusiasm for behavior-changing goals without the skills of reaching them.

Teachers of mathematics are under particular temptation to succumb to this lure of presenting facts for their own sake. This is because a mathematical fact, at least one of the kind commonly taught in the schools, has a peculiar quality of respectability. The

historical fact may be only a lie which history makers once agreed upon and history writers have thereafter copied from one another in solemn erudition. The most substantial fact of chemistry or physics may be modified by new research. But—given the all-important postulates—the mathematical fact stands solidly against all comers. No amount of concerted lying or skillful propaganda can alter it. Research employs it but does not presume to question it—after the postulates are accepted. Its prestige-inspiring qualities are so great that small wonder some of its devotees come to feel that acquisition of so stable a thing is education itself rather than merely a tool of education.

In addition to this danger, teachers of mathematics also face a peculiar opportunity. When they accept the purpose of a genuinely social education and work wholeheartedly toward the achievement of that purpose, they find in their subject an instrument of great educational utility. They can teach mathematics for direct application to a wide variety of practical problems and for indirect application to countless others. They can use it for developing habits of precise generalization. They can employ it to direct the careful thinking in social situations upon which a good society must be based. When they meet this opportunity intelligently, they find their subject expanding and growing more valuable with every practical educational use.

The present revision of a very successful textbook is an excellent discussion of the teaching of secondary-school mathematics from this point of view. The authors are convinced that satisfaction in the historical perfection of mathematics must be replaced by pride in the teaching of educationally needed mathematical skills and concepts. They have written a treatise for teachers and prospective teachers who wish to develop their abilities for this kind of purposeful education. Their wide practical experience and their thorough scholarship in the field of teaching secondary-school mathematics are clearly demonstrated by the effective way they have performed this task.

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PART I

THE PLACE AND FUNCTION OF MATHEMATICS IN SECONDARY EDUCATION

CHAPTER I

THE SECONDARY SCHOOL AS AN EDUCATIONAL UNIT[1]

The American public secondary school, which had its beginning in the third decade of the nineteenth century, was preceded by two other significant institutions of secondary education, *viz.*, the Latin grammar school and the academy. The discrimination between the respective periods for which each of these schools represents the predominant educational pattern is not altogether distinct and clear-cut. The transition from the influence of one educational philosophy to that of another has been characterized by a somewhat deliberate effort at reform under the persistent demands of social change. The development of the new institutions and the gradual fading from prominence of the old have been merely the outward evidences of the efforts of educational leaders to keep abreast of the demands of a changing social order.

The Latin Grammar School. When, in the early part of the seventeenth century, the settlers began the colonization of New England, their customs and institutions were patterned after those of their home country. Thus the immediate prototypes of the early schools of America were the English schools which were patronized primarily by the people of the middle class who needed a knowledge of the Latin language for use in trades and professions.[2] The purpose of the Latin grammar schools in America, the first one of which was established in Boston in 1635, was to prepare boys in Latin grammar and literature for admission to college; and the aim of the colleges was primarily to supply the people with an enlightened clergy.[3]

Record may be found of the existence of such schools in all the

[1] Portions of this chapter originally appeared in *The Mathematics Teacher*, **27** (1934), 117–127, 190–198, 215–244, 281–295, under the title "Development of Mathematics in the Secondary Schools of the United States."

[2] Elmer E. Brown, "The Making of Our Middle Schools" (New York: Longmans, Green & Co., Inc., 1902), p. 7.

[3] E. C. Broome, "A Historical and Critical Discussion of College Admission Requirements" (New York: The Macmillan Company, 1903), p. 24.

colonies before the close of the seventeenth century. In many of the colonies these schools were under private control and were dependent upon tuition or private donations for maintenance. This, however, was not in general the case in New England.[1] In Massachusetts, for example, a law of 1647 ordered that any town of 100 families should establish a grammar school and even went so far as to provide penalties for violation.[2] Although frequently violated, this law, which continued in effect until 1789, brought about more uniformity among the schools of that colony and served as a pattern which was copied in several other colonies. Thus it is seen that the Latin grammar school was a "public school" not in the sense that attendance was free from fees, in accordance with our modern interpretation of "public," but in the sense that it was a "town school," *i.e.*, it was established and controlled by the town as contrasted to the church.

These schools had inherited certain aristocratic characteristics that existed in England at the time the American colonies were settled. These characteristics exhibited themselves in the form of selection of pupils according to the rank and social standing of their parents. Also, under the conditions incident to frontier life, it was inevitable that the schools would be characterized by limited means, limited facilities, limited purposes, and limited opportunities, all of which were manifest in the narrow curriculum of the period. As the population of these colonies increased through immigration and birth, old communities broke up and migration westward began. The new settlements established in the wilderness were founded by people who had not known the religious zeal and oppression of the old country. With this shifting of population there began to develop interests in shipping and commerce as well as in religion and agriculture. It became more and more difficult to maintain the schools by tuition or tax. This social and commercial expansion gradually led to a demand for a more liberal and democratic form of education. An effort to meet this demand resulted in the organization of the academy program of secondary education.

The Academy. As early as 1743 Benjamin Franklin had formulated plans for the establishing of the first academy. The researches of Seybolt have shown, however, that the ideas incorporated in the plan

[1] C. O. Davis, "Public Secondary Education" (Chicago: Rand McNally & Company, 1917), pp. 2–3.

[2] W. H. Small, "Early New England Schools" (Boston: Ginn & Company, 1914), pp. 6–7.

were, in all probability, not altogether original with Franklin.[1] Although the time was ripe for a new plan of education, it was not until 1749 that Franklin was able to gain any significant recognition of his scheme, the educational philosophy of which is best portrayed by the following quotation from his original proposal:

> As to their studies, it would be well if they could be taught everything that is useful and everything that is ornamental. But art is long and their time is short. It is therefore proposed, that they learn those things that are likely to be most useful and most ornamental; regard being had to the several professions for which they are intended.[2]

In 1751 Franklin's Academy opened in Philadelphia with three schools; English, Latin, and Mathematics, each under its own master. In 1754 a fourth school, the Philosophical, was added and this resulted in the reincorporation of the institution in which the Latin and Philosophical Schools were spoken of as the "college" and the other two as the "academy." There were other schools in the middle colonies and farther south, in the third quarter of the eighteenth century, which served the purpose of academies No legislative record can be found, however, to indicate that any such institution, with the exception of the one at Philadelphia, was chartered before the Revolutionary War.[3]

Although the academy was semiprivate in its administration, most states assisted in its founding and support. However, despite this encouragement by public funds, tuition charges were the almost invariable rule.[4] Thus the existence of such schools depended very greatly upon their ability to attract students. This situation resulted in a large variety of offerings and a consequent broadening of the curriculum. Furthermore, "educational planners for the nation proposed to throw off denominational control of education, emphasized unhampered scientific research, and upheld the unfettered right of exposition, while cherishing a deep sense of social responsibility."[5]

[1] I. L. Kandel, "History of Secondary Education" (Boston: Houghton Mifflin Company, 1930), pp. 168–170.

[2] From Paul Monroe, "Principles of Secondary Education" (New York: The Macmillan Company, 1914), p. 54. By permission of The Macmillan Company, publishers.

[3] Brown, *op. cit.*, p. 190

[4] Paul Monroe, "Cyclopedia of Education" (New York: The Macmillan Company, 1911), Vol I, pp 22–23.

[5] Charles A Beard, "The Unique Function of Education in American Democracy" (Washington: Educational Policies Commission, National Education Association, 1937), p. 17.

It was largely through the influence of the academy that geography was recognized for college entrance in 1807; English grammar, in 1819; algebra, in 1820; geometry, in 1844; and ancient history, in 1847.[1] Along with this enforced expansion in subject matter there went a certain amount of experimentation in teaching, which caused a normal-school atmosphere to develop around some of the academies. Another innovation of this new system of education was the extension to girls of the opportunity for attending school. The level of instruction was more of a "secondary type," based on a previous elementary-school training and designed for preparation for college and life, than was the case in the early colonial schools. The academy, to a certain extent an offspring of Philistinism, helped to foster new interests in education; to meet the growing needs of a new country and a new age; and, through its residential character, to break down the barriers of provincialism.[2]

The changing social ideals and demands, which had caused the downfall of the Latin grammar school and had given impetus to the growth of the academy, ultimately began to uncover the shortcomings of this new form of secondary education. The academy was largely privately owned and operated and, consequently, was somewhat dependent upon tuition fees; a fact which, to a very considerable extent, limited the services which it was able to render. Public sentiment soon began to demand a form of secondary education that would more nearly reach the masses. This meant an educational organization supported and controlled by the people, *i.e.*, supported by public taxation and under state supervision.

The Public High School. This movement for reform in secondary education crystallized in 1821 with the establishment of the English Classical School, the name of which was changed to the English High School in 1824. The purpose of this school was to offer a finishing course for boys preparing for mercantile or mechanical vocations as well as to provide preparatory training for those planning to enter college. The curriculum of this new school was strikingly similar to the one offered at the same time in the English department of the Philips Exeter Academy.[3]

The American public did not grasp very readily the concept of a

[1] Kandel, *op. cit.*, p. 420.

[2] L. V. Koos, "The American Secondary School" (Boston: Ginn & Company, 1927), p. 26.

[3] Elmer E. Brown, Secondary Education, *Annual Report of the Department of the Interior* (Washington: Government Printing Office, 1903), Vol. I, p. 563.

national scheme of free secondary education. The old European idea that instruction beyond the "three R's" was meant only for the aristocracy appeared to predominate in the average American's educational philosophy. Although previous legislative endorsement had been given the idea of free education for all,[1] Massachusetts passed a law in 1827 which proved to be the pattern for subsequent laws concerning public secondary education. In this law provision was made for the establishment of educational facilities for the benefit of all inhabitants of "every city, town, or district containing five hundred families," with additional provisions to be made in "every city, or town, containing four thousand inhabitants."[2] By the time of the Civil War the public high school had developed to such an extent that it was contesting rather successfully the predominance of the academy. The hardships incident to the Civil War and the period immediately following gave added impetus to the growing demand for the public promotion of high-school facilities.

By 1890, a well defined movement for reform was gaining momentum under the leadership of President Eliot of Harvard. Out of this new movement, beginning with the Report of the Committee of Ten in 1893, came the reorganization of American secondary education. This famous report, in spite of its conservative recommendations, aroused great interest in secondary education throughout the nation. The thirty years following the issuance of the report saw a series of reports of committees of the National Educational Association and its affiliated organizations dealing specifically with reforms in secondary education. The most important of these were the Committee on College Entrance Requirements (1899), the Committee on Six-year Courses (1905–1909), the Committee on Economy of Time (1905–1913), the Committee on the Articulation of High School and College (1911–1912), and the Commission on the Reorganization of Secondary Education (1912–1922). These several reports, the most farreaching being those of the Commission on Reorganization of Secondary Education, recommended reforms which have changed the form and spirit of the American secondary school from a conventional institution for general education for a small and select portion of the population to a comprehensive institution aiming to educate all the youth of the ages twelve to eighteen or twenty.[3]

During the early stages the movement for reorganization of sec-

[1] Kandel, *op. cit.*, p. 123.

[2] From E. D. Grizzell, "Origin and Development of the High School in New England before 1865" (New York: The Macmillan Company, 1923), pp. 86 87. By permission of The Macmillan Company, publishers.

[3] E. D. Grizzell, "American Secondary Education" (New York: Thomas Nelson & Sons, 1937), pp. 40-41.

ondary education centered around the approximately equal division of time devoted to elementary and secondary education. The idea of dividing the six-year secondary school into junior and senior departments did not become a prominent one until the first decade of the present century. About this time the heterogeneity of the increasing school population, together with a new interest in the psychological and sociological aspects of a functional education program and a new interpretation of the aims of American education, began to direct the thoughts of educational leaders to the desirability of defining the two distinct areas of secondary education now known as the "junior high school" and the "senior high school." The philosophy of the junior high school was defined in terms of the four fundamental principles: (1) better articulation between elementary and secondary education; (2) exploration, revelation, and guidance; (3) interpretation of environment; and (4) motivation.[1] In contrast, the primary function of the senior high school was defined to be the provision for the beginning of content specialization and the pursuance of one's aptitudes and interests.

Although there is still by no means unanimous acceptance of the junior-senior-high-school plan of organization, it now appears to have attained a place of permanent and predominating significance in American secondary education.

The Junior College. Another important variant in the reorganization of secondary education has been the development of the junior college. Although, historically, it first served primarily only as the first two years of the ordinary four-year period of college education, there has been from the beginning a growing sentiment that its program should be definitely integrated with that of secondary education.[2]

The earlier legislative acts and rulings of standardizing agencies rather definitely limited the functions of the junior college to that of

[1] J. M. Glass, Tested and Accepted Philosophy of the Junior High School Movement, *The Junior-senior High School Clearing House*, 7 (1933), 329–339.

[2] J. T. Davis, Overlapping of High School and College Work by Teachers and Students in Junior Colleges, *Proceedings of American Association of Junior Colleges*, 6th Annual Meeting, Chicago, 1926, pp. 16–27.

W. J. Greenleaf, Junior Colleges, *Bulletin* 3, Office of Education (Washington: Government Printing Office, 1936), pp. 3–11.

F. M. McDowell, The Junior College, *Bulletin* 35, Office of Education (Washington: Government Printing Office, 1919), pp 10–70.

H. G. Noffsinger, One-third of a Century of Progress, *The Junior College Journal*, 5 (1935), 395–404.

preparation for more advanced college work.[1] More recently state laws and regulations for standardization have been much more liberal in their interpretations of the objectives of this new educational unit.[2] Today we find that the junior college not only prepares for college, the university, and the technical school, but also provides terminal courses, both vocational and cultural, as well as completion units of secondary education.[3] In place of the single preparatory function of the early days, the junior-college educational program of today is predominantly characterized by four significant functions, *viz.*, (1) the preparatory function, (2) the popularizing function, (3) the terminal function, and (4) the guidance function.[4] Thus we find the junior college taking its place in an educational program designed to provide every qualified individual with that information and those techniques which should better enable him to live a genuinely satisfactory life and make positive contribution to the stability and perpetuity of society.

American Secondary Education. In the three centuries of American secondary education many significant changes have taken place. The one school of 1635, which had only a few boys as students, has expanded into a system of approximately 25,000 public high schools with 6,000,000 students of both sexes, not to mention the private schools with an enrollment of nearly a half-million and the extension of the secondary program into the first two years of college. The aristocratic policy of selecting students from among a favored few has been replaced by the democratic policy of opening the doors of educational opportunity to students from all levels of society. The fundamental philosophy has changed from one which demanded an institution whose principal function was preparation for a single profession to one which calls for an institution whose chief aim is twofold in nature, *viz.*, the provision of terminal training that is functional for all individuals, and the provision of propaedeutic training for those who can, and desire to, derive profit from further study in institutions of higher learning.

The primary business of education, in effecting the promises of American democracy, is to guard, cherish, advance, and make available in the life of

[1] John W. Barton, Trends in the Junior College Curriculum, *The Junior College Journal*, **5** (1935), 405.

Doak S. Campbell, A Critical Study of the Stated Purposes of the Junior College, *Contributions to Education* 70 (Nashville, Tenn.: George Peabody College for Teachers, 1930), p. 56.

[2] Barton, *op. cit.*, pp. 408-409.

[3] Noffsinger, *op. cit.*, p. 403.

[4] Noffsinger, *op. cit.*, pp. 401–404.

coming generations the funded and growing wisdom, knowledge, and aspirations of the race. This involves the dissemination of knowledge, the liberation of minds, the development of skills, the promotion of free inquiries, the encouragement of the creative or inventive spirit, and the establishment of wholesome attitudes toward order and change—all useful in the good life for each person, in the practical arts, and in the maintenance and improvement of American society, as our society, in the world of nations.[1]

As the frontiers of knowledge expand, new social issues will continue to arise to demand fundamental changes in the tenets of the philosophy of American secondary education. The functional educative process will always be that one which strives to stimulate individuals to a continuous reconstruction of their social outlook as well as to prepare them for more intelligent integration with their social enviroment.

Exercises

1. What European influences were most significant in the formation of the educational philosophy of the early settlers?
2. Briefly outline the educational program of the Latin grammar school
3. When and where was the first American secondary school founded? What was its name?
4. Briefly trace the transition from the Latin grammar school to the academy, pointing out the influences that were significant in bringing about this change.
5. When, where, and by whom was the first academy founded? What was its name?
6. Briefly outline its educational program.
7. What specific changes in the educational program of the secondary school were made during the academy period?
8. Briefly trace the transition from the academy to the public secondary school, pointing out the influences that were significant in bringing about this change.
9. Contrast these influences with those that led to the change from the Latin grammar school to the academy.
10. When and where was the first school established that was supported by public taxation and supervised by the state? What was its name?
11. Briefly outline its educational program.
12. What is the approximate date for the division of the secondary education program into the two distinct periods, junior high school and senior high school?
13. Briefly distinguish between the functions of these two divisions of the secondary school.
14. When was the first junior college established?
15. Briefly trace the evolution of the present philosophy of the function of the junior college.

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[1] Beard, *op. cit.*, pp. 77–78.

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CHAPTER II

THE EVOLVING PROGRAM OF SECONDARY MATHEMATICS[1]

Mathematics has not always occupied the same place in the educational program of the secondary school that it occupies today. Instead, it has come to its present status through a long and interesting period of evolution. Since the very beginning of secondary education in this country the offerings of the schools have been influenced by changing educational ideals and changing practical considerations. The place which mathematics has occupied in the instructional program has, to a considerable degree, reflected these ideals and conditions. There have been periods in which its status has been characterized by relative stability and prominence, and there have been other periods in which flux and instability, uncertainty, and depression have been pronounced. One can hardly come to a real appreciation of the present status of mathematical instruction and of its proper function as one of the avenues and instruments of general secondary education without having some picture of this evolution of the mathematical program.

Obviously, such a picture must include some description of the prevailing educational tenets and the practical considerations which have been in evidence at different times and of the effects of these upon the status and character of mathematical instruction. The evolving program of secondary mathematics has been profoundly influenced by the work of certain individuals and by the reports of certain committees. The attitudes of these individuals and committees, their recommendations, and the effects which are traceable to their influence must also form a part of the picture.

Mathematics and the Changing Educational Program. From the time of the first curriculum of the public high school, mathematics has occupied a place of importance in both the elementary and secondary programs. Its prominence has fluctuated considerably, with

[1] Portions of this chapter originally appeared in *The Mathematics Teacher*, **27** (1934), 117–127, 190–198, 215–224, 281–295, under the title "Development of Mathematics in the Secondary Schools of the United States."

the most pronounced rises taking place prior to 1910 and since the impact of World War II, and with the most disturbing recessions occurring during the decade immediately preceding the war.

In the early frontier days the need for mathematics and the motivation for its development were very slender. The farmer and the ordinary worker needed little beyond the ability to add and subtract. Those boys who aspired to enter some form of trade experience needed to know something of simple computation along with a slight knowledge of common measure, a few very simple fractions, and the use of English and other European monies. The seafaring man needed to know the basic principles of navigation; the clergyman found astronomy necessary in fixing the dates of religious festivals; and the public official, when fixing territorial boundaries, found surest recourse in surveying. From this meager source flowed the shallow stream of mathematics in sixteenth-century America. Consequently, it is not surprising that mathematics received little attention in the early elementary schools. They were established for the primary purpose of teaching "writing and reading" with an occasional reference to "ciphering." In fact, one who could qualify as an "arithmeticker" was likely to be considered as especially endowed, although this sobriquet implied no more than that the individual was able to perform the simplest of computations.

The low degree of intellectual effort in these early days caused many of the leading statesmen of the period a great deal of concern. Some made distinct efforts to analyze the situation and to get at the causes. The early settlers were not interested in the works in literature and science produced by the masters of Europe. There was even opposition to public education "on the ground that it made boys lazy and dissatisfied with farm life, and led to religious skepticism."

It was not until the latter part of the nineteenth century that real progress began to take place in the development of mathematics in America. By this time there was a fair-sized group of native talent working in the field of mathematics, and there were major influences shaping up to promote significant development in mathematical research. As most important among these influences we might list the vision of the presidents of certain outstanding universities and their appreciation of the need for promotion of interest in mathematics in this country, the founding of the American Mathematical Society, and a closer contact with European scholars. Many of the young men of that era who were interested in mathematics pursued advanced study and sought advanced degrees in mathematics in European uni-

versities, in particular, those of Berlin, Göttingen, and Leipzig. Furthermore, many European mathematicians were imported to this country to teach in our universities, some temporarily and others on a permanent basis.

It was well that the leaders in the field of mathematics should have become so concerned with the promotion of mathematical research in this country. The research worker is the producer of mathematics. It is he who must stay out on the frontier in the expansion of the effectiveness, usefulness, and significance of the field of mathematics as an important aspect of our culture and social structure. The investments of effort and planning made by these early leaders have paid excellent dividends to the stockholders in American culture, for today the American corps of mathematicians occupy the vanguard of mathematical thought.

Thus we see that, during the course of approximately four centuries, vision and scholarly effort have produced a strong and impressive superstructure of enlightened research in the field of mathematics in our country. But what of the understructure of effective teaching and intelligent subject-matter planning at the elementary and secondary levels? While the events of recent years have created an international feeling of awe and admiration in the contemplation of the marvelous accomplishments of mathematical research, at the same time there have been many pointed expressions of concern over faulty teaching and deficient curricula in mathematics at the elementary and secondary levels.

Modern educational philosophy, to a very large degree, tends to use the child rather than subject matter as the nucleus of all educational procedure. This fact is given strong support in the report on the Eight-year Study which was sponsored by the Progressive Education Association, now known as the American Education Fellowship.[1] This report was the outgrowth of what was termed an "Adventure in American Education" entered into by 30 participating schools. Their search for objectives resulted in the pronouncement that "the chief purpose of education in the United States should be to preserve, promote, and refine the way of life in which we as a people believe."[2] In the perspective of this "great, central purpose" of education the 30 schools proceeded to announce five conclusions pertaining to the curriculum of secondary education.

[1] Wilford M. Aikin, "The Story of the Eight-year Study" (New York: Harper & Brothers, 1942).

[2] *Ibid.*, p. 133.

First, *every student should achieve competence in the essential skills of communication—reading, writing, oral expression—and in the use of quantitative concepts and symbols.*

Second, *inert subject-matter should give way to content that is alive and pertinent to the problems of youth and modern civilization.*

Third, *the common, recurring concerns of American youth, should give content and form to the curriculum.*

Fourth, *the life and work of the school should contribute, in every possible way, to the physical, mental and emotional health of every student.*

Fifth, *the curriculum in its every part should have one clear, major purpose.* That purpose is to bring to every young American his great heritage of freedom, to develop understanding of the kind of life we seek, and to inspire devotion to human welfare.[1]

Whether or not the above report presents a true picture of the prevailing philosophy of modern education, it does emphasize the important fact that the American pattern of a democratic form of education for all American youth creates problems cut from cloth entirely different from that of the schools of the early frontier days. This is given still further emphasis in the following statement from the Educational Policies Commission:

Schools should be dedicated to the proposition that every youth in these United States—regardless of sex, economic status geographic location, or race—should experience a broad and balanced education which will (1) equip him to enter an occupation suited to his abilities and offering reasonable opportunity for personal growth and social usefulness; (2) prepare him to assume the full responsibilities of American citizenship; (3) give him a fair chance to exercise his right to the pursuit of happiness, (4) stimulate intellectual curiosity, engender satisfaction in intellectual achievement, and cultivate the ability to think rationally; and (5) help him to develop an appreciation of the ethical values which should undergird all life in a democratic society.[2]

What are the implications of such a philosophy to the subject-matter field of mathematics? Teachers of mathematics must concern themselves with the content of the instructional materials which they use. No one can, nor should anyone desire to, argue for a mathematics program that is sterile and functionless. If mathematics can make no contribution to the more effective living in our modern complex society, then it has no place in the school curriculum. The good

[1] *Ibid.*, p. 138.

[2] Educational Policies Commission, "Education for *All* American Youth" (Washington: National Educational Association of the United States, 1944), p. 21.

teacher of mathematics will strive at all times to keep a proper balance between the social aims and the mathematical aims which should shape the pattern of his instructional program. There are six major objectives of instruction which delineate very sharply his responsibilities in his efforts to direct the learning activities in which his pupils engage. They are (1) proficiency in fundamental skills; (2) comprehension of basic concepts; (3) appreciation of significant meanings; (4) development of desirable attitudes; (5) efficiency in making sound applications; (6) confidence in making intelligent and independent interpretations. In the framework of these objectives the teacher of mathematics must keep his thinking critically responsive to curriculum trends, and to their implications for mathematics, in the matrix of the prevailing philosophy of education.

It is a far cry from the arithmetic of the Latin grammar school to the varied mathematical program of the modern period. The many changes which have taken place have been motivated by the prevailing tenets of the different philosophic periods through which education has passed. They are reflected in the works of committees and in the effects of various other influences which have left their imprint upon the curriculum. An analysis of these influences and the works of such committees will help one better to orient the function of mathematics in the perspective of the secondary curriculum.

Early Arithmetic. Arithmetic, the only mathematical study of importance in the secondary schools of the United States during the colonial period, made its entrance to the curriculum of the Latin grammar school through the early writing school. Later, in the eighteenth century, we find a tendency toward more general recognition of the value of arithmetic as a school subject. This was due partially to the rise of commercial interests and also to the tendency of the people to be more hospitable toward a subject not supported by tradition.[1] The subject matter of the course was logical in arrangement and consisted primarily of a series of rules to be memorized and dogmatically applied according to rather definite classifications. No attempt was made to adapt the instruction to the pupil, and very little change was made in the nature or content of the subject matter. Under such a system of instruction the only arithmetic studied was "ciphering" which consisted largely of the manipulation of integers.

Textbooks were not generally used until the latter part of the eighteenth century. The master dictated the problems to be solved

[1] Elmer E. Brown, "The Making of Our Middle Schools" (New York: Longmans, Green & Co., Inc., 1902), p. 134.

and stated the rule, or rules, to be used. No special directions or explanations were given. The rules and problems were recorded in "cipher books." In describing two cipher books used in the study of arithmetic in the state of Illinois between 1804 and 1808, Breslich states that the subject taught was but little more than a mechanical manipulation of figures and a study of rules dogmatically applied.[1]

The majority of the arithmetics published during the colonial period were English editions. In the period from 1662 to 1776 there were only 27 arithmetics published, and 9 of these contained a great deal of material foreign to the subject. These texts were written primarily from the utilitarian point of view. Wingate, the author of "The Arithmetick, Containing a Plain and Familiar Method for Attaining the Knowledge and Common Practices of Common Arithmetick," which was published in 1689, stated that the purpose of his arithmetic was "for the Ease and Benefit of such Learners, who desire only so much Skill in Arithmetick, as is useful in Accompts, Trades, and such like ordinary Employments."[2] On the other hand, the second edition (1797) of Pike's "New and Complete System of Arithmetic," which was used in colleges as well as in secondary schools, contained 516 pages divided as follows: 16 pages devoted to Recommendations, Prefaces, Table of Contents, and Explanation of Characters; 396 to arithmetic; 4 to plane geometry; 11 to plane trigonometry; 46 to mensuration of surfaces and solids; 33 to "an introduction to algebra designed for the use of Academies"; and 10 pages to an introduction to conic sections.

Probably the most popular text of this early period was "The Schoolmaster's Assistant" by Thomas Dilworth, first published in England in 1744 or 1745. An American reprint of this book appeared in Philadelphia in 1769, and it later enjoyed at least six American revisions.[3] In the treatment of the subject matter no attempt was made to present any formal demonstration, and all rules and definitions were given in the form of questions and answers. As was characteristic of all texts of the colonial period, this book placed great stress upon the memorizing of rules. Cajori in commenting upon the book

[1] E. R. Breslich, Arithmetic One Hundred Years Ago, *Elementary School Journal*, **25** (1924 1925), 664 674

[2] D. E. Smith, The Development of the American Arithmetic, *Educational Review*, **52** (1916), 111.

[3] F. Cajori, "The Teaching and History of Mathematics in the United States" (Washington: Government Printing Office, 1890), p. 14.

says: "The whole book is nothing but a Pandora's box of disconnected rules. It appeals to memory exclusively and completely ignores the existence of reasoning powers in the mind of the learner."[1]

By the methods of colonial times each problem was solved as being one of a certain type or as belonging to a particular case, and few pupils were able to solve a problem unless they knew under which case it came. This system and these methods of attack were applied to problems on a variety of subjects. In notation and numeration, Dilworth gave rules for numbers up to 9 digits. Pike extended these rules to any number of digits, giving a specific example for a number of 42 digits. While Dilworth treated of the fundamental operations in a number of cases, several other texts presented this material in a much more simplified form. In common, or vulgar, fractions Dilworth presented the different operations as special cases but did not solve an example or illustrate a rule. Under the discussion of decimal fractions he included, in addition to the fundamental processes, the rule of three, interest, discount, equation of payments, and a number of other applications of percentage. The subject of denominate numbers was considered quite important, and Dilworth gave tables for English money; Troy, avoirdupois, and apothecaries' weight; time; and motion. Percentage included problems of loss and gain, discount, and interest; Pike added commission, brokerage, partial payments, buying and selling stock, and policies of insurance. Moreover, both Pike and Dilworth gave an elementary treatment of progressions and permutations, and they discussed such topics as square root, cube root, barter, fellowship, alligation, practice, and position. Pike devoted 30 pages to a group of miscellaneous questions and problems, each problem, or group of problems, being preceded by a rule for the solution. A typical rule is the one for finding two numbers, having given the sum of the numbers and the sum of their squares.

From the square of their sum take the sum of their squares: then from the sum of their squares take this remainder, and the square root of the difference will be the difference of the numbers. To half their sum add half their difference, and the sum will be the greater. From half their sum take half their difference, and the remainder will be the less.[2]

Mental Discipline. The curriculum of the academy was much broader in its scope than was that of the Latin grammar school and,

[1] *Ibid.*, p. 16.
[2] Nicolas Pike, "A New and Complete System of Arithmetic" (Worcester, Mass.: Isaiah Thomas, 1797), p. 360.

in general, less dominated by college-entrance requirements. An examination of the annual reports made to the Regents of the University of the State of New York reveals that many different subjects appeared on various academy curricula during the period from 1787 to 1870, among them the following mathematical subjects: arithmetic, algebra, astronomy, bookkeeping, conic sections, civil engineering, plane geometry, analytic geometry, leveling, logarithms, mapping, mensuration, navigation, nautical astronomy, statistics, surveying, and trigonometry.[1] The particular attention which was given to mathematics during this period was due in part to the possibility of practical applications but principally to the idea of mental discipline which occupied a very prominent place in educational thought during these early years of American education and exerted a profound influence upon curriculum making and methods of teaching. The attitude prevalent in the minds of textbook writers and teachers of this period was well stated by Joseph Ray:

> The object of the study of mathematics is twofold—the acquisition of useful knowledge and the cultivation and discipline of the mental powers. A parent often inquires "Why should my son study mathematics? I do not expect him to be a surveyor, an engineer, or an astronomer." Yet the parent is very desirous that his son should be able to reason correctly, and to exercise, in all his relations in life, the energies of a cultivated and disciplined mind. This is, indeed, of more value than the mere attainment of any branch of knowledge.[2]

The more extreme views on mental discipline did not go unchallenged, especially in the Middle West. The pioneer life made it necessary for the individual to rely upon his ability to develop the natural resources of this new country, and this called for a type of education which placed more emphasis on the utilitarian values than on the disciplinary or the cultural values Although the number of mathematical subjects in the curriculum remained fairly uniform throughout the life of the academy, the nature and purpose of instruction underwent considerable change.

Foreign Influences. The English influence predominated in the American schools from the time of the Revolution until 1820. Many of the instructors in the American colleges and academies had received

[1] From Paul Monroe, "Principles of Secondary Education" (New York: The Macmillan Company, 1914), p 54. By permission of The Macmillan Company, publishers.

[2] Joseph Ray, "New Elementary Algebra" (New York: American Book Company, 1848), p iii

their training in England, and the majority of the texts were either English editions or copied rather closely from English authors. Hutton's "Mathematics," Bonnycastle's "Algebra," and Playfair's "Euclid" are three of the most frequently mentioned texts of this early period.[1]

The period of French influence may be dated from the appointment of Claude Crozet, who had been trained in the École Polytechnique of Paris, as professor of mathematics at the United States Military Academy in 1817. For a period of approximately fifty years from this date texts by French authors were the most popular for general use, although they never entirely displaced the older English texts. John Farrar, who was selected for the Chair of Mathematics and Natural Philosophy at Harvard, in 1818 published an "Introduction to the Elements of Algebra," selected from the "Algebra" of Euler, and a translation of Lacroix's "Algebra"; he also published a translation of Lacroix's "Arithmetic."[2] Charles Davies, professor of mathematics at the United States Military Academy (1823–1837), published Brewster's "Translation of Legendre" (1828) and Bourdon's "Algebra" (1834).[3]

Contemporaneous with the influx of French mathematics there was a revival of interest in elementary education under the influence of the Pestalozzian school. The first incorporation of Pestalozzian ideas in the preparation of an arithmetic text was in Warren Colburn's "Intellectual Arithmetic upon the Inductive Method of Instruction," otherwise known as the "First Lessons."[4] This text, published first in 1821, marked the beginning of a new epoch in arithmetic and the teaching of arithmetic. The pupil was now supposed to replace the studying and memorizing of rules that he did not always fully understand by the setting up of his own rules as generalizations of his actual experiences. He was introduced to new topics by means of practical problems and questions, the order of presentation always being from the concrete to the abstract.[5] Colburn published a more advanced book in 1822 as a "sequel" to his "First Lessons." This text was designed as a practical arithmetic to be studied after the more elementary text. It consisted of topics usually found in the earlier advanced arithmetics except for

[1] Cajori, *op. cit.*, pp. 55–56.
[2] *Ibid.*, p. 128.
[3] *Ibid.*, p. 120.
[4] *Ibid.*, p. 106.
[5] Walter S. Monroe, Development of Arithmetic as a School Subject, *Bulletin* 10, Office of Education (Washington: Government Printing Office, 1917), pp. 63–70, 80–88.

certain omissions—in particular, the rule of three, rule of position, and powers and roots. The method of presentation was the same in the two texts.[1]

The period from 1821 to 1857 was one of rapid development for the study of arithmetic. One hundred and ninety-five texts on this subject were published during these years; of this number, 46 were designed for use in colleges and academies and 4 were direct translations from the French.[2] During the period from 1860 to 1892 there was no essential change in aim or content and few modifications in the methods of teaching.[3] In contrast with the previous period, however, there was significant change in the organization and presentation of subject matter. The arithmetics of this latter period were combinations of the old, as found in colonial arithmetics, and the new, as found in the works of Colburn. The old arithmetics generally rejected reasoning; Colburn's arithmetics rejected rules and encouraged reasoning; the texts of the period 1860–1892 gave rules but at the same time gave demonstrations and encouraged pupils to think.

The Pestalozzian influence for reform in content selection and instructional technique was felt in algebra and geometry as well as in arithmetic. Warren Colburn, who published his "Introduction to Algebra upon the Inductive Method of Instruction" in 1832, advanced new ideas as to methods of instruction in algebra. His idea was to make the transition from arithmetic to algebra as gradual as possible. As the learner was expected to derive most knowledge from solving the problems himself, the explanations were made as brief as were consistent with giving what was required. The problems were designed to exercise the learner in reasoning instead of making him a mere listener.[4] Geometry, before the nineteenth century, was taught dogmatically, and the students merely memorized and worked by rule. Although there remain traces of the dogmatic method of instruction, since the middle of the nineteenth century there has been a gradual trend toward more emphasis on original exercises and construction problems with more attention given to intuitive geometry.

[1] Walter S. Monroe, Analysis of Colburn's Arithmetics, *Elementary School Teacher*, **13** (1912–1913), 239-246.

[2] J. M. Greenwood, American Textbooks on Arithmetic, *Annual Report of the Commissioner of Education* (Washington: Government Printing Office, 1897–1898), Vol. I, pp. 796–868.

[3] Walter S. Monroe, Development of Arithmetic as a School Subject, *Bulletin* **10**, Office of Education (Washington: Government Printing Office, 1917), p. 90.

[4] H. G. Meserve, Mathematics One Hundred Years Ago, *The Mathematics Teacher*, **21** (1928), 339.

The Public High School. It was not until the last quarter of the nineteenth century that the public high school predominated over the academy. Rapid changes were being made in the social, political, and industrial customs of the United States, and evidence of the attempts of the high school to keep pace with these developments may be seen in the large number of courses added to the curriculum. By 1860 both algebra and geometry had become firmly established in the curricula of the state high schools of Massachusetts. At this time we find that 45 of a selected list of 63 schools also offered surveying; 15 offered navigation and mensuration; 37 offered trigonometry; and 1 school listed analytic geometry.[1] As in the academy there was a strong tendency toward expansion both in number and content of courses offered, as well as an attempt to blend intellectual and practical training in the same school. Curricula were organized and expanded rapidly with no particular plan or definite educational objective in view. By 1890 this unrest had reached its highest point.[2] This awkward and unsystematic expansion of the curriculum offered sufficient reason for a demand for reform, and mathematics received its share of the attack. Dissatisfaction had arisen from several sources relative to the results achieved in the teaching of secondary mathematics. Complaints had come from the teachers of mathematics themselves that the subject was not being grasped by the pupils. A study of a large number of representative high schools had revealed that the largest percentages of failures were in Latin and mathematics.[3] College faculties were not hesitant in letting it be known that students entered their freshman classes with poor mathematical training. Businessmen were doubtful of the opportunity for the application of high-school mathematics, as taught, to problems of everyday life.

The Committee of Ten. The Committee of Ten on Secondary School Subjects agreed that a radical change in the teaching of mathematics was necessary. The Subcommittee on Mathematics recommended that a course in concrete geometry with numerous exercises be introduced into the grammar school and that systematic algebra should be begun at the age of fourteen. It was suggested that demon-

[1] A. J. Inglis, The Rise of the High School in Massachusetts, *Contributions to Education* 45 (New York: Bureau of Publications, Teachers College, Columbia University, 1911), pp. 110–116.

[2] I. L. Kandel, "History of Secondary Education" (Boston: Houghton Mifflin Company, 1930), p. 461.

[3] F. P. O'Brien, The High School Failures, *Contributions to Education* 102 (New York: Bureau of Publications, Teachers College, Columbia University, 1919), p. 21.

strative geometry should follow the first year of algebra, that it should be taught along with algebra for the next two years, and that work in solid geometry might be incorporated. Formal algebra was to be studied for 5 hours a week during the first year and for 2½ hours a week for the two following years, during which time it was to parallel work in geometry. Special emphasis was to be placed on literal as well as numerical coefficients. The Committee also suggested that those who did not expect to go to college might, after the first year of algebra, turn to bookkeeping and the technical parts of arithmetic, while boys planning to attend scientific schools might profitably spend a year on trigonometry and some more advanced topics of algebra. A hope was expressed by the Committee that a place might be found in the high-school or college course for at least the essentials of modern synthetic or projective geometry.[1]

The Committee on College Entrance Requirements. In order to bring about a better articulation between the secondary schools and colleges the Committee on College Entrance Requirements in 1899 recommended the following course:

Seventh Grade: Concrete Geometry and Introduction to Algebra; Eighth Grade: Introduction to Demonstrative Geometry and Algebra; Ninth and Tenth Grades: Algebra and Plane Geometry; Eleventh Grade: Solid Geometry and Plane Trigonometry; Twelfth Grade: Advanced Algebra and Mathematics reviewed.

Algebra for the seventh and eighth grades was to begin with literal arithmetic which was to be followed by simple polynomials and fractional expressions, equations of the first degree with numerical coefficients in one and two unknowns, the four fundamental operations for rational algebraic expressions, and simple factoring. One-half of the time of the seventh grade was to be devoted to concrete geometry, while in the eighth grade one-half of the time was to be spent in demonstrative geometry. The important objective of such work was to awaken an interest in demonstrative geometry. It was recommended that an equal amount of time be devoted to algebra and geometry in the ninth and tenth grades.[2]

The International Commission. Another influence for change in the curriculum of secondary mathematics was that due to the reports of

[1] *Report of the Committee of Ten on Secondary School Subjects* (New York: American Book Company, 1894), pp. 105–116.

[2] A. F. Nightingale, Report of the Committee on College Entrance Requirements, *Proceedings and Addresses of National Education Association*, 38th Annual Meeting, 1899, pp. 648–651.

the International Commission on the Teaching of Mathematics which were published by the United States Bureau of Education between the years 1911 and 1918. Committee III of the Commission, in its study of "Mathematics in the Public General Secondary Schools of the United States," found that every high school offered algebra and geometry for at least one year each. One-half of the schools gave algebra for an extra half-year and less than twenty per cent gave algebra for the full two years. There were very few schools that offered algebra for two and one-half years and only the larger high schools had courses in solid geometry, plane trigonometry, and advanced algebra.[1]

The sequence in practically all the textbooks in geometry was that of Legendre. Geometrical constructions by the Euclidean method[2] were usually given a logical place among other propositions. The original exercises, which ranged from 600 in one text to 1,200 in another, were rigidly confined to the subject matter of the text.

In addition to subjects usually recognized as secondary school subjects, several high schools offered a course in commercial arithmetic similar to the course frequently given in the elementary school except that the problems were somewhat more difficult and more closely related to commercial life.[3]

In their analysis of the report of Committees III and IV, the

[1] Report of Committees III and IV of the International Commission, Mathematics in the Public and Private Secondary Schools of the United States, *Bulletin* 16, Office of Education (Washington. Government Printing Office, 1911), pp. 17–22.

[2] The first three postulates stated by Euclid in his famous "Elements" are· "Let it be granted,

"*a*. That a straight line may be drawn from any one point to any other point:

"*b*. That a terminated straight line may be produced to any length in a straight line:

"*c*. And that a circle may be described from any centre, at any distance from that centre."

The first two of these postulates enable us to draw a straight line, but not lines of a prescribed length except insofar as a line might be drawn to connect two given points In other words, these two postulates permit all operations possible with an unmarked straightedge. The third postulate allows the use of the compasses for the drawing of a circle with a given center and passing through a given point, *i e.*, with a fixed radius.

Thus "geometrical constructions by the Euclidean method" are those constructions which can be performed using only the unmarked straightedge and compasses.

For further discussion see Hilda P Hudson, "Ruler and Compasses" (New York: Longmans, Green & Co., Inc., 1916).

[3] *Ibid.*, p. 22.

American Commissioners of the International Commission found the following very marked tendencies for change in curriculum and method:

1. To omit geometric proofs that are either obvious or too difficult
2. To transfer the more difficult portions of the algebraic matter hitherto given in the first year of the high school to a later year
3. To avoid algebraic manipulations of greater complexity than is requisite to prepare pupils thoroughly for the work that lies beyond
4. To give more prominence to the equation
5. To introduce more problems from physics and other sciences and from practical life
6. To modify the conception of the aim of the teaching to conform to what is understood to be the outcome of recent psychologic research concerning the value of "formal discipline"
7. To attach greater importance to the utilitarian possibilities of mathematics[1]

After their general survey of the field of secondary mathematics the Commission felt that there were two main needs which were dominant, namely,

> The need for the better preparation of teachers and the need to reduce, if not eliminate, the waste of effort involved in independent and often inadequate treatment of fundamental and broad questions by separate schools, colleges, or local systems.[2]

In their study of elementary mathematics in the college the Commission found that calculus had become primarily a sophomore study and that it was serving somewhat as "a boundary line between two styles of teaching," the recitation method and the lecture or lecture-quiz method. This difference in the educational problem before and after calculus was further emphasized by the proposal, on the part of some, to relegate the first two years of college work to the high school and, on the part of others, to take care of this work in "a junior college leading to an appropriate degree." The organization of subject matter was still largely compartmentalized, but there was developing a tendency toward fusion with more emphasis on practical applications. The general outline of material was algebra, trigonometry, analytic geometry, and calculus with a prerequisite of elementary algebra through quadratics, plane geometry, and sometimes solid geometry. The changing complexion of the school population was

[1] Report of the American Commissioners of the International Commission on the Teaching of Mathematics, *Bulletin* 14, Office of Education (Washington: Government Printing Office, 1912), pp. 29–31.

[2] *Ibid.*, pp. 39–40.

recognized as a cause for readjustment of content and instructional technique in the freshman year.[1]

The National Committee of Fifteen on Geometry Syllabus. In 1908 the Mathematics Round Table of the Secondary Department of the National Education Association unanimously called for a committee to study and report upon the problem of a syllabus for geometry. During the same year the American Federation of Teachers of the Mathematical and Natural Sciences authorized the appointment of such a committee, and in 1909 the Secondary Department of the National Education Association authorized the committee to proceed as a joint committee of the Association and Federation.

In its report[2] the Committee recommended that reasonable attention be given to concrete exercises but with no diminishing attention to the logical structure of geometry. It suggested that there be "a quickening of the logical sense" through a distribution of emphasis designed to economize on the time and energy spent in the mastery of theorems and to provide more time and opportunity for the study of geometry in its more concrete relations. It pointed out that there are some terms in geometry which it is best to accept as undefined and recommended that definitions be introduced at the time when needed rather than massed at one place in the text or course of study. The desirability of the use of certain informal proofs was mentioned as well as the advisability of excluding limits and incommensurables from the requirements for entrance to college. It was recommended that between one and one and one-half years be given to plane geometry and that it be taught simultaneously with algebra or preceded by at least one year of algebra.

In the treatment of exercises it was recommended that careful thought be given to their distribution and gradation. The Committee seemed to feel that the tendency had been to overemphasize difficult abstract applications of various theorems in the exercises. It, therefore, suggested concrete exercises along with "a judicious selection of a reasonable number of abstract originals." Material for exercises was to be found in other subject fields as architecture, natural

[1] *Ibid.*, pp. 41–47.

Report of Committee XII of the International Commission, Graduate Work in Mathematics in Universities and in Other Institutions of Like Grade in the United States, *Bulletin* 6, Office of Education (Washington: Government Printing Office, 1911), pp 45–47.

[2] Committee of Fifteen, Provisional Report on Geometry Syllabus, *School Science and Mathematics*, 11 (1911), 330.

design, indirect measurement, and any other source available to the individual teacher. A great deal of emphasis was given to the nature and importance of the concept of locus.[1]

The National Committee on Mathematical Requirements. In 1916 the National Committee on Mathematical Requirements was organized under the auspices of the Mathematical Association of America, Inc., "for the purpose of giving national expression to the movement for reform in the teaching of mathematics, which had gained considerable headway in various parts of the country, but which lacked the power that coordination and united effort alone could give."[2]

The Committee was instructed to concern itself with making a comprehensive study of the whole problem of mathematical education on the secondary and collegiate levels. One of the major problems was thus to make proper provisions for the comparatively new, yet fairly well-established, junior-senior-high-school program of instruction. In its final report the Committee formulated the aims of mathematical instruction into three general classes: practical, disciplinary, and cultural.[3] Instead of a detailed syllabus for the junior high school the Committee proposed only a general outline by topics and stated that further experimentation was necessary before determining a standardized syllabus. Although the Committee refused to commit itself upon the arrangement of topics, it did suggest five plans for distribution of this material, hoping that they might prove helpful in the organization of mathematics curricula for the junior high school. It further recommended that the mathematics proposed for the grades of the junior high school be required of all pupils.[4]

The Committee recommended that provision be made for the specific aims of the mathematics of the senior high school through a body of elective material which should be open to all pupils who had satisfactorily completed the required work of the junior high school. Fully realizing that the method of organization of this material could be more elastic in nature than that for grades seven, eight, and nine, and also thoroughly cognizant of the fact that no one best plan had been determined, the Committee suggested four different plans, any one of which might be used for the purpose of more efficiently organ-

[1] *Ibid.*, pp. 434–460, 509–531.

[2] The National Committee on Mathematical Requirements, "The Reorganization of Mathematics in Secondary Education" (Boston: Houghton Mifflin Company, 1923), p. vii.

[3] *Ibid.*, pp. 6–13.

[4] *Ibid.*, pp. 29–42.

izing the instructional content of the mathematics of grades ten, eleven, and twelve.

Additional electives such as elementary statistics, mathematics of investment, shop mathematics, surveying and navigation, and descriptive or projective geometry were suggested for schools where there was a need for such work and where the conditions warranted their inclusion in the curriculum. It was also recommended that extensive use be made of historical and biographical material in the entire teaching program to lend interest and significance to the subject matter studied.[1]

General Mathematics. The arrangement of subject matter has always been a problem of at least as much difficulty as has its selection. The early tendency was simply to follow the compartment system of organization of the traditional high-school course. This scheme has given way in part to a plan which has continued to gain favor among educators in general. The new idea of arrangement of material whereby the courses of study were to be more closely related to each other was designated as "correlated," "composite," "unified," "cooperative," "integrated," or more commonly as "general" mathematics. This plan for the organization of mathematics can largely be traced to the efforts of the Committee of Ten and the Committee on College Entrance Requirements during the latter part of the nineteenth century. Although there had been a few scattered instances of such treatment of mathematical subject matter prior to this time, the reports of these two committees seem to carry the first evidence of any concentrated thought devoted to its consideration. In an address before the National Education Association (1902) Newhall expressed the hope that a time would come when the secondary school course would comprise six years and when mathematics would not be limited by artificial boundaries as was the case in the study of algebra, geometry, and trigonometry.[2] Contemporaneous with this address the influence of Klein in Germany, Tannery and Borel in France, Perry and Nunn in England, and Moore in America made the conditions more favorable for the rapid growth of general mathematics.

Numerous experiments have been made to determine the status of this new organization of subject matter. McCormick has summarized the conclusions derived from the most significant of these:

1. There is no very clear or definite agreement among mathematicians and general educators as to what constitutes general mathematics.

[1] *Ibid.*, pp. 48–57.

[2] Charles W. Newhall, Correlation of Mathematical Studies in Secondary Schools, *Proceedings of the National Education Association* (1902), pp. 488–492.

2. General mathematics is gradually replacing the traditional type in the seventh, eighth, and ninth grades.

3. General mathematics provides training for college mathematics that is as good as, and perhaps better than, that of mathematics of the traditional type.

4. The indications are that general mathematics creates more interest in the subject than does traditional mathematics.

5. The opinions of high school teachers reveal a large number of reasons for teaching each type of mathematics. General mathematics is favored by many because of the wide variety of information given, because of the interest created, and because of its practical value. Traditional mathematics is favored by many others because of the more thorough knowledge imparted, because of its better organization, and because of the belief that it gives a better preparation for college.

6. Most of those persons who have specialized in methods of teaching mathematics are in favor of a more general type than has been offered.

7. Attempts are being made to correlate the different branches of mathematics. Textbook writers of today, however, are careful not to try to fuse material when an unnatural correlation results.[1]

As a result of the general mathematics movement in the junior high school we find that the mathematics courses have been changed from the formal compartmental arrangement to studies with concrete beginnings, less scientific rigor, more developmental and explanatory material, more practical exercises, better psychological development, more provisions for individual differences, and better fusion of related topics. This effort to present the mathematics of the junior high school in a fused course has not been without its rather severe critics. There are those who feel that it is too much of a hodgepodge of superficialities that tends to general weakening of the subject content. Some critics have felt that too much emphasis is given to fusion of subject matter and not enough attention paid to the significance of the individuality of different topics. Nevertheless, the fact that two of the fundamental services of the junior high school have been to provide for exploration and for contact with minimum essentials has helped to concentrate interest in general mathematics as an instructional unit in its curriculum.

This effort to evolve a functional mathematical program well adapted to the education of the masses has also brought about significant changes in the content and instructional techniques of senior-

[1] Clarence McCormick, Teaching of General Mathematics in Secondary Schools, *Contributions to Education* 386 (New York: Bureau of Publications, Teachers College, Columbia University, 1929), p. 162.

high-school mathematics. The mathematical content for these later years of secondary instruction is more specialized in nature than that of the three previous years and, as a consequence, has not been so readily adapted to a fused method of organization. One of the most difficult of the problems which confront the teachers of senior-high-school mathematics is that of so organizing and presenting subject content as to preserve its intrinsic characteristics and yet to introduce the desired continuity.

In the development of the mathematics curriculum of the junior college there are, at present, some indications of a trend toward general mathematics for the first year. Many teachers of freshman mathematics feel that students taking such a course get a deeper appreciation of the fundamental interrelations of mathematical material and consequently have a better idea of the real meaning of mathematics than do those who take the traditional compartmentalized type of course. They also claim that such a student will attain just as satisfactory a degree of mastery of the fundamentals. Although there is a great deal of disagreement as to the desirability of such a course for those who intend to pursue their study of mathematics beyond the freshman year, there is rather universal agreement that the general-mathematics type of course is probably the better program for those who plan to take no further work in mathematics.[1]

College-entrance Requirements. The problem of the curriculum of secondary mathematics has always been closely related to the question of college-entrance requirements. The entire program of the Latin grammar school was defined in terms of preparation for college. The academy, however, early expanded its program to include not only preparation for college but also training for life. In meeting the demands of this new function of secondary education, the academy gradually encroached upon the educational program of the colleges. The natural consequence of this was a material increase in college-entrance requirements.

At the close of the eighteenth century the only mathematics required for admission to college was "a knowledge of the rules and processes of vulgar arithmetic." The statutes of 1807 prescribed that the requirements for admission to Harvard should include the rules of arithmetic

[1] J. S. Georges, Mathematics in the Junior College, *School Science and Mathematics*, 37 (1937), 302–316.

P. C. Scott, A Comparative Study of Achievement in College Freshman Mathematics, *Contributions to Education* 243 (Nashville, Tenn.: unpublished Ph.D. thesis, George Peabody College for Teachers, 1939).

dealing with simple and compound notation, addition, subtraction, multiplication, division, reduction, and the single rule of three. In 1820 these requirements were extended to include the algebra of simple equations, roots, and powers, arithmetical and geometrical progressions. Columbia added algebra to her entrance requirements in 1821; Yale, in 1847; and Princeton, in 1848. In 1844 candidates for admission to Harvard College were examined in arithmetic, algebra, and geometry (up to the book on proportion). Geometry was made an entrance requirement at Yale in 1865; at Princeton, Michigan, and Cornell in 1868; and at Columbia in 1870. By 1870 the admission requirements at Harvard had been extended to include higher arithmetic, algebra through quadratic equations, logarithms, and the elements of plane geometry. Yale and Princeton specified only to quadratic equations in algebra. While Yale specified the first two books of Playfair's Euclid (or an acceptable equivalent), Princeton specified only the first book. Similar requirements existed at the other leading universities of that period.[1]

Since 1890 the trend has been toward the definition of college-entrance requirements in terms of one year each of algebra and geometry. While there has been a distinct decline even in the number of colleges that insist upon these subjects, it is safe to say that the typical modern college-entrance requirement would call for one year of algebra and one year of plane geometry.[2]

The National Committee on Mathematical Requirements in its 1923 report recognized the far-reaching influence of college-entrance requirements upon the teaching of secondary mathematics. It criticized the prevailing type of examination as overemphasizing "the candidate's skill in formal manipulation." In an effort to make desirable modifications in the prevailing type of college-entrance examination, members of the National Committee met with members of a committee from the College Entrance Examination Board and drew up a list of recommendations which were incorporated in the 1923 report. While attention was paid to the problem of reducing the excessive "difficulty and complexity of the formal manipulative questions," two of the most significant recommendations were as follows:

[1] E. C. Broome, "A Historical and Critical Discussion of College Admission Requirements" (New York: The Macmillan Company, 1903), pp. 41–53.

[2] Benjamin Fine, "Admission to American Colleges" (New York: Harper & Brothers, 1946), pp. 3, 30.

H. C. McKown, Trend of College Entrance Requirements, *Bulletin* 35, Office of Education (Washington: Government Printing Office, 1924), p. 65.

1. An effort should be made to devise [algebraic] questions which will fairly test the candidate's understanding of principles and his ability to apply them, while involving a minimum of manipulative complexity.

2. The examinations in geometry should be definitely constructed to test the candidate's ability to draw valid conclusions rather than his ability to memorize an argument.[1]

Early in 1921 the College Entrance Examination Board appointed a commission to study the problem of college-entrance requirements and to make recommendations as to desirable revisions in the definitions of requirements in elementary mathematics. In their reports published in 1923[2] the Commission eliminated from the list of requirements in algebra the extended and useless manipulation of polynomials, reduced factoring to three types, and simplified the requirements in fractions. Increased recognition was given to the formula and the graph, and two notable changes were made in the simplification of the material dealing with surds and in the introduction of numerical trigonometry. For plane geometry only 89 theorems were included in the syllabus; of these only 31 were starred to be used for purposes of proofs. In the case of the unstarred propositions the candidate was expected to be familiar with their content so that he might answer questions related to their substance or use them in solving originals. Similar provisions were made for solid geometry. A new type of examination was one designed to take care of a one-year course in plane and solid geometry. The Committee made the effort to reorganize the examination material in such a way that the demands on the candidate's memory would be lightened and that increased opportunity for the development of geometrical understanding would be given. These examinations have functioned to set up more definite goals of instruction as well as to provide a principal pattern of textbook construction. There are those who feel that the influences have been more detrimental than helpful because of the enforced curriculum, the restricted selection of materials and content, and the curbing of creative work on the part of teachers.[3] It is true, however, that the work of the Board has helped to clarify the problem of entrance requirements and to provide for better articulation between the work of the senior high school and the freshman year of college.

[1] The National Committee on Mathematical Requirements, *op. cit.*, pp. 76–77.

[2] College Entrance Examination Board, *Documents* 107 and 108 (New York: College Entrance Examination Board, 1923).

[3] L. H. Whitcraft, Mathematics and the College Entrance Examination Board, *Contributions to Education* 557 (New York: Bureau of Publications, Teachers College, Columbia University, 1933), pp. 104–106.

In 1935 the Commission on Examinations in Mathematics, a commission of the College Entrance Examination Board, completely revised the type of examinations in mathematics. In these new examinations, known as "alpha," "beta," and "gamma," the effort has been made "to combine the advantages of the longer essay-type, multiple-step question with those of the single-step question."[1] Two of the significant differences between the new and the old form of entrance examinations are to be found in the increased objectivity in scoring and the reduced emphasis upon the traditional compartmentalized treatment of subject matter.

The Commission postulated that "the examinations should be such as to determine: (*a*) the candidate's understanding and appreciation of the fundamental principles and characteristic modes of approach of mathematics; (*b*) his technical equipment and his knowledge of mathematical facts."[1] Furthermore, in recognition of the different interpretations of the meaning of fitness for college, the Commission attempted to provide examinations for the three following groups:

(α) Those who are not ready to carry on in college the study of mathematics or natural science, but who base their claim to be admitted to college in part upon the study of mathematics in the secondary schools

(β) Those who intend to fulfill at least the minimum college requirement in mathematics or natural science

(γ) Those who look forward to more advanced undergraduate work in mathematics and science[2]

In presenting these new examinations the Commission stated that while, in specifying the scope of the examinations, it desired to exercise sufficient definiteness to avoid uncertainty on the part of the teacher, yet it was "strongly influenced by the wish to leave teachers of mathematics in secondary schools free to guide the development of their pupils in such ways as seem to them most desirable."[3]

Soon after the introduction of these examinations there began to develop a rather general demand from schools that there be provided an examination suitable for candidates whose first two years had been devoted primarily to the study of algebra. In response to this demand the Committee of Examiners in Mathematics of the College Entrance

[1] College Entrance Examination Board, "Description of Examination Subjects" (New York: College Entrance Examination Board, 1940), pp. 34–37. An earlier draft of this document, later edited, was published in *The Mathematics Teacher*, **28** (1935), 154–166.

[2] *Ibid.*

[3] *Ibid.*

Examination Board introduced in June, 1942, *Mathematics 2A* (*Alternative Alpha*). Since 1943 mathematics testing by the Board has gone through five distinct but unsatisfactory phases, each emphasizing comprehensive and scholastic aptitude type tests.[1]

National Organizations. No discussion of the forces that have had significant influence in the evolution of the program in secondary mathematics would be complete without mention of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics.

The Mathematical Association of America, Inc., was organized at Columbus, Ohio, in December, 1915, and was incorporated under the laws of the state of Illinois on September 8, 1920. In the interest of improved instruction in collegiate mathematics the Association has sponsored the following:

1. The publication of *The American Mathematical Monthly*, a high-grade mathematical magazine devoted to the interests of collegiate mathematics.
2. The organization of a large number of sections where papers in mathematical research are presented and instructional problems in collegiate mathematics are discussed.
3. The organization of many undergraduate clubs in colleges and universities. These clubs have been very effective in motivating interest in mathematics.
4. The appointment of many committees for the study of problems related to the content and methods of mathematical instruction and the better training of teachers of mathematics. Some of the committees have been independent committees of the Association; others have been joint committees with representation from other interested groups.

In addition to the sectional meetings the Association holds two regular national meetings each year at which papers are read and instructional problems are discussed. Further evidences of the interest of the Association in the promotion of mathematics are as follows: (1) annual subsidies paid to journals that are interested in the publication of mathematical research; (2) the publication in December, 1929, of "The Rhind Mathematical Papyrus"; (3) the publication of the *Carus Monograph Series;* and (4) the awarding of the $100 Chauvenet Prize, the purpose of which is to stimulate expository contributions in mathematical journals.

[1] College Entrance Examination Board, *Forty-ninth Annual Report of the Director* (New York: College Entrance Examination Board, 1949), pp. 7-9.

The National Council of Teachers of Mathematics was organized at Cleveland, Ohio, on February 24, 1920, and was incorporated under the Illinois laws on April 28, 1928. The purpose of the Council is to promote interest in mathematics, especially in the elementary and secondary fields, by the following means:

1. The holding of meetings for the presentation and discussion of papers. At the present time four such meetings are held each year: one in the winter, one in the spring, and two in the summer. One of the summer meetings is held in conjunction with the summer meeting of the National Education Association, of which the Council, while retaining its identity and independent status, became a department on July 8, 1950.

2. The publication of a journal, books, papers, and reports for the purpose of vitalizing and coordinating the work of local organizations of teachers of mathematics and of bringing the interests of mathematics to the attention and consideration of the educational world.

The official organ of the Council, *The Mathematics Teacher*, contains articles by leading mathematicians and outstanding classroom teachers. It is the only magazine in America whose interests are devoted entirely to the better teaching of mathematics on the elementary and secondary levels. A further significant contribution to the literature of better instruction in mathematics is the *Yearbook*, published at intervals since 1926 by the Council, which is devoted to the discussion of important aspects of the teaching of elementary and secondary mathematics.

A new venture in the interest of mathematics and its place in the modern educational program is the semipopular monograph series projected by the Council. The purpose of this series of publications is to enlighten the public as to the nature and significance of mathematics as well as to enrich the instructional background of the teacher of elementary and secondary mathematics. The first of these monographs, "Numbers and Numerals," by Jekuthiel Ginsburg and David Eugene Smith, was published in 1937.

3. The promotion of the affiliation of local organizations of teachers of mathematics with the Council and of close cooperation with other professional organizations. An extension of this plan for the creation of interest and the development of group and national consciousness has been the establishing of state representatives in each state of the Union and a central office through which contacts are kept. Recently the Council has become somewhat international in nature through such affiliation with groups from certain provinces of Canada.

4. The promotion of investigations for the purpose of improving the teaching of elementary and secondary mathematics. This work has largely been carried on through the sponsoring of committees such as the Committee on Individual Differences,[1] the Committees on Geometry,[2] the Committee on Arithmetic,[3] the Joint Commission to Study the Place of Mathematics in Secondary Education, the Commission on Post-War Plans, and a permanent Research Committee.

The Central Association of Science and Mathematics Teachers, Inc., although a sectional organization, has exerted a national influence over the teaching of secondary mathematics. This has been accomplished in the main through the publication of *School Science and Mathematics*, "a journal for all science and mathematics teachers."

In this connection there should be mentioned also the *Mathematics Magazine*, published in Pacoima, Calif. This magazine devotes a portion of each issue to "The Teaching of Mathematics," a section in which are discussed teaching problems as they are primarily related to the junior college.

The Joint Commission to Study the Place of Mathematics in Secondary Education. In the fall of 1933 a Commission to Study the Place of Mathematics in Secondary Education was appointed by the Mathematical Association of America, Inc. Later this Commission was incorporated into a Joint Commission of the Mathematical Association and the National Council of Teachers of Mathematics. In its final report[4] the Commission undertook to define the place of mathematics in the modern educational program and then to organize a mathematical curriculum for grades 7 to 14 (secondary education was defined to include the junior college) in terms of the major mathematical fields which would provide for continuity of development and flexibility of administration.

[1] Report of Committee on Individual Differences, *The Mathematics Teacher*, **25** (1932), 420–426; **26** (1933), 350–365.

[2] The reports of the different committees on geometry may be found in *The Mathematics Teacher*, **24** (1931), 298–302, 370–394; **25** (1932), 427–428; **26** (1933), 366–371; **28** (1935), 329–379, 401–450.

[3] The National Committee on Arithmetic, Arithmetic in General Education, *Sixteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1941).

[4] Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, *Fifteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1940). (Hereafter referred to as the Joint Commission.)

It is the opinion of this Commission that the obvious difficulty of providing for both continuity and flexibility has been the great stumbling block in the development of a nation-wide mathematical program of instruction. Accordingly, in this Report is described a program for mathematics in grades 7 to 14 that definitely aims to provide for continuity of development, and that at the same time respects the reasonable demands for flexibility on the part of school administrators and teachers.[1]

The proposed program was based upon an assumed "normal mathematical equipment of the American pupil who has satisfactorily completed the work of the sixth grade," which was defined as follows:

1. A familiarity with the basic concepts, the processes, and the vocabulary of arithmetic
2. Understanding of the significance of the different positions that a given digit may occupy in a number, including the case of a decimal fraction
3. A mastery of the basic number combinations in addition, subtraction, multiplication, and division
4. Reasonable skill in computing with integers, common fractions, and decimal fractions
5. An acquaintance with the principal units of measurement, and their use in every day life situations
6. The ability to solve simple problems involving computation and units of measurement
7. The ability to recognize, to name, and to sketch such common geometric figures as the rectangle, the square, the circle, the triangle, the rectangular solid, the sphere, the cylinder, and the cube
8. The habit of estimating and checking results[2]

In recognition of existing differences in educational philosophies and practices and in an effort to make "a definite step toward educational harmony," the Commission set up a tentative list of guiding principles to be followed in organizing the mathematical program for grades 7 to 12. In the light of these principles two classifications of the materials of mathematical instruction were made.

1. *The subdivision according to major subject fields*
   - I. The field of number and computation
   - II. The field of geometric form and space perception
   - III. The field of graphic representation
   - IV. The field of elementary analysis
   - V. The field of logical (or "straight") thinking
   - VI. The field of relational thinking
   - VII. The field of symbolic representation and thinking

[1] *Ibid.*, p. 53.
[2] *Ibid.*, p. 54

2. *The subdivision according to certain broad categories enumerated as follows:*
   I. Basic concepts, principles, and terms
   II. Fundamental processes
   III. Fundamental relations
   IV. Skills and Techniques
   V. Applications[1]

In Chaps. V and VI of the report the Commission proposes and discusses in considerable detail two alternative curriculum plans, differing somewhat in detail and emphasis but both giving recognition to the guiding principles enumerated and both consonant with the foregoing dual classification of subject matter. For one of these plans a Grade Placement Chart displays a suggested allocation and organization of the subject matter in each year for grades 7 through 12.[2]

The report contains suggestions for modification of the program to give flexibility to the curriculum. It contains also a discussion of the problems of both retardation and acceleration in their bearing upon the program of mathematical instruction. In this connection there is presented a second Grade Placement Chart giving a proposed selection and grade allocation of subject matter for slow pupils in grades 7, 8, and 9.[3]

In the discussion of the mathematics program for grades 13 and 14 (the junior college) the Commission points out that the preparatory type of student is in the minority and that the mathematical needs of this type of student are adequately provided for in the traditional courses. Accordingly, the discussion deals mainly with the problem of providing for the terminal type of student. Two programs, somewhat different in nature, are proposed as possible terminal courses for those students who do not plan to pursue the study of mathematics further. These courses are based on the Commission's belief that the junior-college curriculum should make provision for a mathematical course of at least one year in extent for all students.

The Commission emphasizes that in no sense is this recommended program of mathematical instruction to be regarded "as a final unchanging yardstick inhibiting personal initiative and further experimentation." The intention is, rather, that this might prove to be a step forward in securing "for mathematics the place in education it so richly deserves" and a safe standard for comparison in this era of curriculum experimentation and change.

[1] *Ibid.*, p. 61.
[2] *Ibid.*, pp. 72–119, 246–251.
[3] *Ibid.*, pp. 120–148, 252–253.

The Progressive Education Association Committee on the Function of Mathematics in General Education. In 1932 the Executive Board of the Progressive Education Association established the Commission on Secondary School Curriculum. This Commission subsequently established several Committees to explore the respective contributions of various subject fields to general education at the secondary level. Among these was the Committee on the Function of Mathematics in General Education. The complete report of this Committee was issued in tentative (mimeographed) form in 1938 and was published in final form in 1940.[1]

The report is in four parts. The first of these presents the educational philosophy which guided the Committee in the formulation of the report. Central in this philosophy is the premise that mathematics, in order to justify its place in the secondary-school curriculum, must contribute to the satisfaction of the needs of the students. These needs are enumerated in terms of the following four "basic aspects of living":

1. Personal Living
2. Immediate Personal-social Relationships
3. Social-civic Relationships
4. Economic Relationships[2]

This part of the report closes with a discussion of the role of mathematics in satisfying the needs of people with respect to these four aspects of living.

Part II is the most extensive section of the report. It consists of an elaborate discussion of certain broad concepts or understandings which find application in problem solving whether this be in situations that are peculiarly mathematical or not. In the discussion of these concepts an effort is made to show their applications to situations encountered in ordinary living as well as to strictly mathematical situations. In other words, much emphasis is placed upon the *generality* of these concepts and upon various broad aspects involved in their understanding. Separate chapters are devoted to the consideration of the following major concepts:

[1] Commission on Secondary School Curriculum of the Progressive Education Association, Mathematics in General Education, ***Report of the Committee on the Function of Mathematics in General Education*** (New York: Appleton-Century-Crofts, Inc., 1940).

[2] *Ibid.*, p. 20.

Formulation and Solution
Data
Approximation
Function
Operations
Proof
Symbolism

Part III is concerned with an explanation of the nature of mathematics and of its development.

Part IV considers the problem of understanding the student and stresses the need for considering not only overt behavior but also the various influences which have operated to shape the student's personality. It contains a section on implications for teaching. The report closes with a chapter on the evaluation of student achievement. This chapter discusses the purposes of evaluation and contains some interesting suggestions of new and unique means and devices for organizing an evaluation program in terms of instructional objectives previously set up.

The nature of the report as a whole is indicated in its introductory chapter. It confines itself mainly to the discussion of a program of mathematical education in terms of broad outlines and general principles and does not attempt to set forth any detailed organization of subject matter. It recognizes frankly that the formulation of a series of courses based upon its proposals would require years of experimentation and that such experimentation would probably eventually modify some of the suggestions made in the present report. It looks to the future instead of trying to set up a practical program for the immediate present.

> It is with respect to this point that the task undertaken by this Committee differed from that of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics. In preparing its report of *The Place of Mathematics in Secondary Education*, the Joint Commission, after discussing the general aims of education, sought to outline a program of the sort being offered at the moment by some schools in advance of the great majority. Most of its suggestions have been tested to some extent in practice, and the Joint Commission took a practical rather than experimental point of view.[1]

The AAAS Co-operative Committee on the Teaching of Science and Mathematics. Concomitant with the great increase in size in the program of secondary education in the United States during the last

[1] *Ibid.*, p. 14.

three centuries there has been a vast expansion of purpose. Modern educational problems are rooted in the context of mass education, which seems to nurture extensive, and at times seemingly excessive, curriculum design These problems will always remain complex; there is no way to simplify them. The only hope for an intelligent approach to their solution is through the cooperative effort of duly organized groups who are interested in the formulation and promotion of that program in our schools which will be most significant when evaluated in terms of the educational needs of a school population that is highly heterogeneous as to abilities, aptitudes, and interests.

An important step in this direction was made in 1941 when the representatives of several scientific societies created the Co-operative Committee on Science Teaching "to work on educational problems the solution of which can be attained better by co-operative action than by any single scientific group working alone." The Mathematical Association of America was one of the societies represented on the original Committee before its reorganization, in 1944, as a committee of the American Association for the Advancement of Science (AAAS). As presently constituted, the Committee consists of representatives from 14 national scientific societies, two of which are the Mathematical Association of America and the National Council of Teachers of Mathematics.

Since its organization the Committee has given attention to the problem of the preparation of teachers of science and mathematics. In 1946 it published its recommendations relative to this important problem in a report on The Preparation of High School Science and Mathematics Teachers, in which the following proposals were made:

(1) A policy of certification in closely related subjects within the broad area of the sciences and mathematics should be established and put into practice.

(2) Approximately one-half of the prospective teacher's four-year college program should be devoted to courses in the sciences.

(3) Certificates to teach general science at the 7th-, 8th-, or 9th-grade level should be granted on the basis of not less than 42 semester hours of college courses in the subjects covered in general science.

(4) Colleges and certification authorities should work toward a five-year program for the preparation of high-school teachers.

(5) Curriculum improvements in the small high school should go hand in hand with improvement in teacher preparation.[1]

[1] Report of the AAAS Co-operative Committee on the Teaching of Science and Mathematics, The Preparation of High School Science and Mathematics Teachers, *School Science and Mathematics*, **46** (1946), 107–118.

The Committee not only has directed its attention to many other problems and projects but it also has projected a rather active future program of study and service.[1] Among the most important projects undertaken was the study of the effectiveness of the teaching of science and mathematics at all levels, which served as the basis for a report incorporated as Appendix II of Volume 4 of the report on *Science and Public Policy* made by the President's Scientific Research Board.[2] In this study appraisals of the science and mathematics programs from grades 1 through 12 were made. These served as the basis for certain recommendations for improvement, among which were the following which pertained to mathematics:

(1) A complete appraisal should be made of science and mathematics teaching in secondary schools. This should include a survey of curriculum offerings, student enrollments, available laboratory and demonstration equipment, methods of instruction, the workweek of the teachers, and the total preparation of teachers for their responsibilities.

(2) The secondary-school mathematics curriculum should be studied to determine the effectiveness of present offerings with regard to the general education needs of all students and the special needs of students talented in science and mathematics. New courses in mathematics should be designed wherever indicated by this study.

(3) The secondary-school science curriculum should be reorganized so as to permit . . . at least 3 years of science for students with special talents in science and mathematics . . .

(4) . . .

(5) Studies should be made concerning the place, value, and effective use of biographical and historical materials relating to science and mathematics.

(6) Studies should be made of the various curricular and administrative arrangements employed in small and large communities to meet the needs of talented youth. Reports should be prepared to make more teachers acquainted with best practices.

(7) Studies should be made of the guidance procedures used in secondary schools. . . . Bulletins revealing effective materials and practices concerning science and mathematics should be prepared for and studied by secondary-school teachers and counselors.

(8) Studies should be made to determine the most effective ways to use demonstration, laboratory, project, shop, and field experiences and such facilities as library materials, audio-visual aids, etc., in the teaching of science and mathematics.

[1] The Co-operative Committee on the Teaching of Science and Mathematics: Its Organization and Program, *Science*, **106** (July 11, 1947), 28–30.

[2] President's Scientific Research Board, *Science and Public Policy* (Washington: Superintendent of Documents, Government Printing Office, 1947), **4**, 47–149.

(9) A study should be made of the administrative devices which will encourage greater use of community resources in the teaching of science and mathematics.

(10) The work of science and mathematics supervisors, special consultants, visiting teachers, and other special advisory personnel should be studied with a view to making more prevalent the practices and techniques most effective in developing sound programs involving science and mathematics for general education, and in providing optimum opportunities for the development of students with special talents.[1]

The report also included many very pertinent recommendations for the recruitment and training of teachers of science and mathematics in the elementary and secondary schools.[2]

The Emergency of World War II. During the past quarter of a century there has been an increasing amount of evidence accumulating which points to the inadequacy of the mathematics program in our schools. This inadequacy became very conspicuous in the light of the deficiencies in mathematics discovered among the inductees into the war-training program of World War II. To meet the emergency of the situation the U.S. Office of Education in cooperation with the National Council of Teachers of Mathematics appointed two committees to give the problem careful study. The first committee worked in close cooperation with the Army, Navy, and Civil Aeronautics Administration. Its report[3] was based on a detailed analysis of approximately 50 unit courses used in the Federal-state program of Vocational Training for War Production Workers, 20 Navy training manuals, and 50 Army instructional manuals. The Committee gave attention to the entire program of secondary mathematics and made specific recommendations for a Special One-year Course and a Special One-semester Course designed as an emergency refresher course for high-school pupils near graduation or induction but not studying mathematics.

The report[4] of the second committee served as an extension of that of the first committee in that it was designed to supplement the earlier report "by amplifying the suggestions offered for the *lower* levels of mathematics as represented in the Special One-year Course." The

[1] *Ibid.*, pp. 93-94.

[2] *Ibid.*, pp. 107–109.

[3] National Council of Teachers of Mathematics, Report on Pre-induction Courses in Mathematics, *The Mathematics Teacher*, **36** (1943), 114–124.

[4] National Council of Teachers of Mathematics, Report of the Committee on Essential Mathematics for Minimum Army Needs, *The Mathematics Teacher*, **36** (1943), 243–282.

procedure used by this committee consisted of conferences "with Army officers directly in charge of training enlisted men" and observation of "the basic training process itself during the first thirteen weeks of the inductee's Army life." A rather detailed outline of minimum essentials was given along with a presentation of "general suggestions with respect to instruction." Although this report was directed entirely to the wartime emergency, its summary of the "minimum essentials" has definite implications for civilian needs.

> . . . young men about to enter the Army must be taught . . . *the ability to meet quantitative problems effectively, confidently, and sensibly. They must be able (a) to identify the quantitative aspects of the situations which confront them, (b) to deal with these situations by approximation and estimation when computation is not required, (c) to recognize and use the simpler symbolisms of mathematics, (d) to tell when and how mathematical symbolism, concepts, and processes are to be employed, and (e) to compute accurately, quickly, and intelligently when computation is called for.*[1]

These two reports are given special prominence because of their specific and important implications concerning the teaching of mathematics in the elementary and secondary schools. The War Preparedness Committee (appointed in 1940) and the War Policy Committee (appointed in 1943) were joint committees of the American Mathematical Society and the Mathematical Association of America. Their interests were primarily in the organization and direction of man power in mathematical research for most efficient and effective service to the war effort. A similar committee was the National Committee of Physicists and Mathematicians, appointed in 1943. Some of the reports of these committees had significant import for mathematics at the secondary level.[2]

The Commission on Post-War Plans. At its annual meeting in February, 1944, the Board of Directors of the National Council of Teachers of Mathematics created the Commission on Post-War Plans for the purpose of planning for effective programs in secondary mathematics in the postwar period. As originally constituted the Commission consisted of 5 members. This number was expanded later to

[1] *Ibid.*, p. 246.

[2] William L. Hart, On Education for Service, *The American Mathematical Monthly*, **48** (1941), 353–362. Also published in *The Mathematics Teacher*, **34** (1941), 297–304.

Marston Morse and William L. Hart, Mathematics in the Defense Program, *The Mathematics Teacher*, **34** (1941), 195–202.

William L. Hart *et al.*, Universal Military Training in Peace Time, *The Mathematics Teacher*, **39** (1946), 17–23.

include 13 members from 11 widely scattered states. The Commission found significant background for its thinking and planning in the work of the first two of the above-mentioned committees of the war period.

In its First Report[1] the Commission announced its plan of organization and solicited help from the interested public in these words: " . . . we are asking discerning school people and thoughtful laymen for good ideas and definite suggestions from which a sensible report may later stem that will provide adequate training in mathematics for *all* students in our schools—each according to his needs." With the hope of helping to crystallize thinking and "in order to promote discussion . . . in workshops and professional courses" the Commission proposed, and discussed briefly, the following five recommendations which it felt "should be carefully considered by many persons in planning our work for the postwar years:"

1. *The school should insure mathematical literacy to all who can possibly achieve it.*
2. *We should differentiate on the basis of needs, without stigmatizing any group, and we should provide new and better courses for a high fraction of the schools' population whose mathematical needs are not well met in the traditional sequential courses.*
3. *We need a completely new approach to the problem of the so-called slow learning student.*
4. *The teaching of arithmetic can be and should be improved.*
5. *The sequential courses should be greatly improved.*

The Commission's Second Report[2] contained its recommendations for the improvement of mathematics in grades 1 to 14. These recommendations were presented in the form of 34 theses. The first thesis stated that: "*The school should guarantee functional competence in mathematics to all who can possibly achieve it.*" Realizing the desirability of being explicit as to the implications of "functional competence in mathematics," the Commission proceeded to delineate its essentials in the form of a Check List consisting of 28 items. This list was later expanded in the "Guidance Pamphlet"[3] to include 29 items. While full significance of the Check List can be had only by reference to the questions listed in it, some idea of its implications can be

[1] Commission on Post-War Plans, First Report, *The Mathematics Teacher*, **37** (1944), 226–232.

[2] Commission on Post-War Plans, Second Report, *The Mathematics Teacher*, **38** (1945), 195–221.

[3] Commission on Post-War Plans, Guidance Report, *The Mathematics Teacher*, **40** (1947), 315–339. Later published as the "Guidance Pamphlet in Mathematics."

obtained from the following list of key ideas or concepts of each of the 29 items:

1. Computation
2. Per cents
3. Ratio
4. Estimating
5. Rounding numbers
6. Tables
7. Graphs
8. Statistics
9. The nature of a measurement
10. Use of measuring devices
11. Square root
12. Angles
13. Geometric concepts
14. The 3-4-5 relationship
15. Constructions
16. Drawings
17. Vectors
18. Metric system
19. Conversion
20. Algebraic symbolism
21. Formulas
22. Signed numbers
23. Using the axioms
24. Practical formulas
25. Similar triangles and proportion
26. Trigonometry
27. First steps in business arithmetic
28. Stretching the dollar
29. Proceeding from hypothesis to conclusion[1]

The remaining theses of the Second Report were presented according to the following outline:

I. *Mathematics in Grades One to Six.*

Theses 2 through 8 give emphasis to the consideration of arithmetic as a content subject as well as a tool subject, and as having both mathematical and social aims; the need for more careful attention to meanings and wiser use of drill; the realization of the importance of readiness and the futility of incidental teaching; and desirability for more careful evaluation of learning.

II. *The Mathematics of Grades Seven and Eight.*

Theses 9, 10 and 11 point out the uniqueness of the demands of the unified program for all students for these two grades.

III. *Mathematics in Grade Nine.*

Theses 12 and 13 emphasize the need for a double track program in mathematics and also for a careful consideration of the content of the algebra program in the ninth grade.

IV. *Mathematics in Grades Ten to Twelve.*

Theses 14 through 20 treat of the main objectives and essential characteristics of the sequential courses of these three years in the high-school program.

V. *Mathematics in the Junior College.*

Theses 21, 22, and 23 call attention to the different demands likely to be made on the mathematics program of the junior college by three different groups of students: those with cultural interests only, those with pre-vocational needs, and those who have major interests in mathematics.

[1] *Ibid.*, pp. 318–319.

VI. *The Education of Teachers of Mathematics.*

Theses 24 through 32 discuss in fairly full detail some of the significant problems of teacher preparation at the various levels of instruction.

VII. *Multisensory Aids in Mathematics.*

The last two theses of the report briefly emphasize the significance of multisensory aids in effective instruction in mathematics at all levels.

In 1942 the Consumer Education Study was organized with two chief purposes in mind: (1) to investigate what should be taught and how it could best be organized and objectively presented; (2) to facilitate the work of the schools by providing instructional materials. It was recognized that mathematics had an important contribution to make to consumer education, both in the elementary school and in the secondary school. The Commission on Post-War Plans was invited to make recommendations as to what this contribution might and should be. In 1945, in conjunction with the Consumer Education Study, the Commission published a pamphlet entitled *The Rôle of Mathematics in Consumer Education.*[1] In this report the Commission first analyzed the nature and purposes of consumer education in general. It then proceeded to point out the relation of elementary and secondary mathematics to this total program and also to suggest to administrators and teachers a form for organization of materials as well as methods of instruction.

One of the recognized major problems in modern secondary education is that of intelligent guidance of pupils as they plan their school programs in the context of the actualities of the present and the horizons of the future. It is a well-established fact that in no subject-matter area have high-school students suffered more from erratic and unwise guidance than in the field of mathematics. The Commission on Post-War Plans made a significant attempt to correct this unfortunate situation in the publication of the previously mentioned "Guidance Pamphlet," in the preparation of which they were aided materially by the counsel of representatives from the U.S. Office of Education, who were men of wide experience in the problems and techniques of educational guidance.

The pamphlet was addressed to the high-school student in the spirit suggested by this opening paragraph:

[1] Commission on Post-War Plans, *The Rôle of Mathematics in Consumer Education* (Washington: The Consumer Education Study, National Association of Secondary-School Principals, 1945).

Why should I study mathematics? What good will mathematics be to me? Perhaps you have asked yourself these two questions. If so, you have a right to good answers, and you will find them in the following pages.

The Commission then proceeded in simple style to present significant information for the high-school student within the framework of the following carefully determined outline, from which it suggested that the reader should select only those areas in which he is interested:

I. Mathematics for Personal Use
II. Mathematics Used by Trained Workers
III. Mathematics for College Preparation
IV. Mathematics for Professional Workers
V. Women in Mathematics
VI. Mathematics Used by Civil Service Workers
VII. Mathematical Organizations
VIII. Graduate Schools Offeiing the Doctorate in Mathematics
IX. Selected References on Mathematical Careers

Mathematics in General Education. The demands of a program in "general education" are now making their imprint upon the curriculum from the elementary school through the junior college. Such demands are to be interpreted as the fruition of desires that the educational program shall be designed so that it will fit young people "for those common spheres which, as citizens and heirs of a joint culture, they will share with others."[1] In a sense this is an age-old problem in a modern frame of reference. The function of education is to cultivate that "union of knowledge and reason in an integrated personality" which fosters life as a cultured human being and a responsible citizen and, also, provides competencies in areas of specialized interest. We are now living in an age that demands specialism. These demands have tended to shape the curricula of the secondary schools and colleges for the training of experts. The claim has been made that too often the college graduate of today "is 'educated' in that he has acquired competence in some particular occupation, yet falls short of that human wholeness and civic conscience which the co-operative activities of citizenship require."[2] There are many who would use these words also to characterize the graduate of the modern secondary school. The trend toward specialization has led to the introduction of many

[1] The Harvard Committee, "General Education in a Free Society" (Cambridge, Mass.: Harvard University Press, 1945), p. 4.

[2] President's Commission on Higher Education, Report of, "Higher Education for American Democracy," Vol. I, "Establishing the Goals" (Washington: Superintendent of Documents, Government Printing Office, 1947), p. 48.

courses and varied curricula both in the secondary school and in the college. The net result of such emphases has been a decreasing opportunity on the part of the student to participate in a program from which he might derive an "integrated view of human experience." As an antidote to this trend there has been evolving a plan of general education which emphasizes "those phases of nonspecialized and nonvocational learning which should be the common experience of all educated men and women."[1]

What is the contribution which mathematics can make to such a program? This is a question that demands and deserves careful thought. There is a body of mathematical content that is of significance to every individual capable of intelligent participation in the educational program, whether it be at the level of the elementary school, secondary school, or college.

> Numbers are . . . an important means of communication. We call mathematics into service in our daily lives much more frequently than is generally supposed. General education must provide a functional knowledge of the elements of mathematics that industrial society normally requires, and also the skill of quantitative thinking.[2]

There have been several commendable efforts already at defining "a functional knowledge of the elements of mathematics." Among the most significant of these are the previously mentioned reports of the Commission on Post-War Plans. Current literature carries frequent articles portraying the viewpoints of individuals. Betz has presented a convincing synopsis of the pattern of mathematics in the framework of American education in his "three main components of functional competence in mathematics":

> 1. . . . the systematic study, within a desired range, of the underlying mathematical concepts, principles, skills, and modes of thinking.
> 2. . . . a proper emphasis on the significant interrelations between mathematical theory and its many-sided applications.
> 3. Functional competence in mathematics is largely the outgrowth of a continuous and painstaking emphasis on the categories of understanding, mastery, and transfer.[3]

The total effect of this growing influence on the program in mathematics, of course, remains to be seen. There are three vital places in the curriculum where it might affect materially the program and

[1] *Ibid.*, p. 49.
[2] *Ibid.*, p. 53.
[3] William Betz, Functional Competence in Mathematics—Its Meaning and Its Attainment, *The Mathematics Teacher*, 41 (1948), 195–206.

methods of instruction in mathematics, *viz.*, the junior high school, the senior high school, and the junior college. It would seem that the full implications of a program in general education upon the role of mathematics as an educational medium can be properly appraised only through the cooperative efforts of authoritative groups well informed both in the demands of a program of education in a free democracy and in the contributions which mathematics can make to such a program.

What of the Future? The evolving program of mathematics in the secondary schools of the United States can be traced very accurately through a careful analysis of the studies and reports of the committees that have been active. One of the significant trends in recent years has been toward cooperative thinking on common problems. Is it without reason to raise the question as to the possibility that we have reached that point in our educational growth when the entire mathematical program in our schools needs to be subjected to careful reconsideration? Should not such a study be made by a cooperative group of educators composed of those trained in the field of mathematics and those competently familiar with the administrative and instructional problems of a democratic program of mass education? Such a study could be most effective in setting up reasonable expectations in mathematical attainment if it were undertaken on a basis of professional cooperation for the purpose of designing an educationally significant program in mathematics patterned in the interest of all the children in our schools.[1]

Exercises

1. What have been some of the more important aspects of the evolving philosophy of education in the United States?

2. Cite evidences of the effect each of these points of view has had on the mathematics curriculum.

3. Criticize or justify the "six major objectives" given on page 16. Do they or do they not outline the full instructional responsibility of the teacher of secondary mathematics?

4. What was the mathematics curriculum of the Latin grammar school?

5. How did this change in the academy?

6. At approximately what time did textbooks begin to be used extensively in mathematical instruction?

7. Name four important mathematics texts of the Latin-grammar-school period and also four of the academy period.

8. Briefly trace the influence of college-entrance requirements in mathematics on the secondary mathematics curriculum.

[1] See Howard F. Fehr, A Proposal for a Modern Program in Mathematical Education in the Secondary Schools, *School Science and Mathematics*, **49** (1949), 723–730.

9. What evidences are there of French and English influences in the instruction in mathematics during these early periods?

10. In what way did the Pestalozzian movement affect the teaching of mathematics?

11. Name the early texts that most vividly reflected this influence.

12. Name four texts that played a very important part in shaping the early instruction in algebra and geometry.

13. Name six influences that have been significant in the evolution of the mathematics curriculum of the secondary school. Give the major contribution of each.

14. What was the purpose of the Committee of Fifteen on Geometry, and what were its main recommendations?

15. Briefly outline the recommendations for secondary mathematics made in the report, "The Reorganization of Mathematics in Secondary Education."

16. What classification of aims was given in the report, "The Reorganization of Mathematics in Secondary Education"?

17. Point out the distinguishing characteristics of each group of aims. Also indicate the extent to which the groups overlap each other.

18. Briefly outline the recommendations of the Joint Commission concerning the program of secondary mathematics.

19. Contrast these recommendations with those of the National Committee on Mathematical Requirements.

20. What do you understand by the general-mathematics movement? By what other names has this movement been designated?

21. Compare the value of general mathematics in the senior high school with its value in the junior high school and in the junior college.

22. In what ways has the junior-college movement affected the curriculum of secondary mathematics?

23. What is your evaluation of the Joint Commission's discussion of the role of mathematics in civilization and its place in the educational program?

24. In what respects does the Progressive Education Association Committee's Report differ from the Report of the Joint Commission?

25. Compare the two reports with respect to their recommendations on evaluation in mathematics.

26. What are some of the more significant aspects of the nature and work of the AAAS Co-operative Committee on the Teaching of Science and Mathematics?

27. Criticize or justify the statement that the italicized quotation on page 44 has definite implications as to the minimum essentials in mathematics for civilian needs.

28. Briefly outline the work of the Commission on Post-War Plans.

29. Criticize or justify the Check List of functional competence in mathematics as presented in the Second Report and "Guidance Pamphlet."

30. Briefly summarize the most important of the implications of the 34 theses of the Second Report.

31. Point out significant uses that may be made of the "Guidance Pamphlet."

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CHAPTER III

MATHEMATICS AS A FUNCTIONAL PART OF THE SECONDARY CURRICULUM[1]

The American system of public education has been developed to assist in perpetuating, improving, and realizing significant democratic ideals. Such ideals envisage democracy as a desirable mode of living. The school therefore has an obligation to direct the students in the cultivation and control of those competencies, attitudes, or types of behavior which will make for living in harmony with the democratic theory of social life. Such a program requires that the functional relationships of the individual to society be made the basic point of orientation for the educational program. It implies that the student be acquainted with social realities as well as theory; that he be guided into more effective and extensive participation in the activities of the groups of which he is a part; and that these activities be given meaning and purpose.

At the same time, this program must take cognizance of the fact that the most effective participation in such activities will often require a sounder basis than can be afforded by firsthand social experiences alone. There must also be backgrounds of knowledge against which to project situations, rational thought patterns to analyze them, and skills to control or adapt them. Attitudes and behavior patterns must be rationalized as well as emotionalized. The need to *do* should not be regarded as independent of, or more *important than*, the need to *know;* for behavior is without proper guidance except as knowledge may help to provide such guidance.

Since one of the primary democratic ideals is that of self-realization, this concept of education further implies that the student be given opportunity for attaining the optimum development of his individual potentialities. In much of the current literature on curriculum construction, the social motif predominates to the extent of obscuring any substantial emphasis upon the development of the higher intellectual

[1] Portions of this chapter originally appeared in *Educational Administration and Supervision,* **22** (1936), 354–360, under the title "A Cycloramic View of Mathematical Education."

processes. This is unfortunate, and it is both theoretically and practically inconsistent with the true democratic ideal.

> . . . it is to be pointed out that such (the higher mental) processes are of major importance to the individual who is capable of them and to the race to which the individual belongs. The great rewards of civilization go not to the men who are strong of muscle or swift of foot, but to those who advance knowledge and elevate human attitudes. If by any means the educational system can discover how to promote even in the slightest measure the development of the higher mental processes, great advantages will be gained for civilization.[1]

In a curriculum pointed toward the ends of genuine social and individual competence, mathematics will play an important part. Its concepts, methods, and formulations at the elementary levels are directly useful to the individual in many situations having social or practical bearings. They are indirectly useful to him in countless others. Mathematics provides a medium and an instrument uniquely adapted to the understanding and control of natural and social phenomena, and it affords unexcelled opportunities at all levels for developing the higher mental processes in the form of generalizing relations and applying these generalizations once they have been attained. The role of mathematics as a functional part of the secondary curriculum cannot be regarded as unimportant.

Functional Curricula. The effort to keep the program of the public school abreast of the rapid changes taking place in the social, political, and industrial life of the nation has been continually reflected in the changes that have taken place in the philosophy and content of the school curriculum. Evidence of this is seen in the many changes effected, during the course of recent years, in the organization and presentation of subject matter on the elementary and secondary levels of instruction. In the junior high school this revision of mathematical instruction has been very pronounced. There has been an injection of new significant materials, a deletion of many traditional practices and units of subject matter, a reallocation of concepts and skills, a new emphasis on the abstraction of mathematical processes, and a vitalization of instructional techniques. While there have been scattered curricular changes in the mathematical program in the senior high school and junior college, these changes have been largely confined to a readjustment of traditionally accepted units of instruction with occa-

[1] From C. H. Judd, "Education as Cultivation of the Higher Mental Processes" (New York: The Macmillan Company, 1936), pp. 3–4. By permission of The Macmillan Company, publishers.

sional additions of seemingly significant material. Until of late, there has been no concerted effort to part from the dictates of tradition. Instead, there has been a rather persistent acceptance of a convenient heritage of teachable units of subject matter selected and organized largely in terms of a factual and logical pattern of instruction.

Many of the recent proposals for curriculum reform throughout the entire school program find root in a new educational philosophy. This philosophy, instead of emphasizing primarily the stern disciplines of logically organized subject matter, makes subject matter subservient to emphasis on desirable patterns of conduct and on the development of personality. In a curriculum formulated upon this philosophy, the selection of content will be determined by its functional worth in an instructional program designed to give the student abundant opportunity to understand, interpret, and appreciate the major functions of life as a member of a democratic social order.

> The basic function of education is . . . to make youth better disposed and better able to contribute to the betterment of society, either by participating with their maximum effectiveness in the accepted modes of life or by perceiving other and better modes, which they are active in convincing their fellows are superior.[1]

If mathematics is to make its full contribution in the attainment of such objectives as those outlined under this present-day educational philosophy, teachers of mathematics will have to arouse themselves from a static satisfaction in the historical perfection of their subject to a dynamic realization of the need for the formulation of its content into units of instruction that are vital and timely in their significance.

> . . . the reason subject-matter has lost its once complete hold on education in this country is not that it is subject-matter. It is rather that its formulations are not vital and timely, pertinent to the age and stage of culture in which we are moving ahead.[2]

In an educational program organized in terms of functional curricula which are socially significant, the different educational levels will no longer be unique instructional units in a more or less disconnected series of graduated periods of factual instruction. They will be merely sectors, of social and administrative convenience perhaps, in a continuous educational process that affords to every individual—to

[1] T. H. Briggs, A Philosophy of Secondary Education Today, *Teachers College Record*, **36** (1935), 595.

[2] W. H. Kilpatrick, *et al.*, "The Educational Frontier" (New York: Appleton-Century-Crofts, Inc., 1933), p. 80.

the extent that his ability, maturity, and interests permit—abundant opportunity to understand and interpret his social and physical environment to the end that he may function more efficiently in shaping the form of an emergent society to accord with the American ideals of democracy and personal liberty and dignity.[1]

Mathematics and Functional Curricula. What is the distinctive function of mathematics in such a program?

Mathematical techniques afford a most effective means of investigating, tabulating, classifying, and interpreting natural and social phenomena; mathematical concepts and symbolism provide a rhetoric of concise expression which is elegant in its simplicity and exactness; and mathematical subject matter sets a pattern for logical precision and objective evaluation. The teacher of mathematics is not likely to take issue with the proclamation of the educationist: "aims of education for American schools must be defined in terms of certain generalized controls of conduct which, if developed, will lead to the realization of the democratic ideal."[2] On the other hand, he will point to the fact that many of these "generalized controls of conduct" find their maximum significance only through mathematical interpretation. To list a few:

1. The laws of interdependence operate in the social order, national life, international affairs, and the universe around us.
2. Man in his artistic and architectural creations has recognized that the patterns set by nature are inherently symmetric and essentially geometric.
3. The conclusions reached through any chain of logical reasoning are no truer than their fundamental assumptions and definitions.
4. The laws of probability provide a scientific basis for protection against personal and property disaster.
5. Any science tends to become exact and to lend itself to prediction insofar as it becomes mathematical.
6. The practice of thrift and sound investment is essential to the economic stability and progress of the social order.
7. Security and comfort of structure largely depend upon the geometry of form and the mathematics of stresses and strains.
8. The checking of authority and the verification of results tend to cultivate independence of thought.
9. The habit of following a line of thought through a chain of logical deductions to a verified conclusion is a desirable attribute of character.

[1] Commission on the Social Studies, American Historical Association, "Conclusions and Recommendations" (New York: Charles Scribner's Sons, 1934), p. 39.

[2] H. L. Caswell and D. S. Campbell, "Curriculum Development" (New York: American Book Company, 1935), p. 125.

10. Science and invention have annihilated distance and speeded up communication.

11. A carefully planned budget is necessary to a smoothly functioning domestic or business life.

12. The thought processes involved in making generalizations from the concrete to the abstract are desirable characteristics of an educated individual.

13. Analysis and synthesis are significant characteristics of constructive thinking.

14. The disposition to give sustained concentrated effort to the solution of a difficult problem is very desirable.

15. To be able to evaluate data, to select that which is significant, and to eliminate the superfluous are indispensable techniques of thinking.

16. The graph is unsurpassed as a method of depicting functional and statistical values and relationships.

17. Number is the language of science.

18. The discovery of the principle of position, or place value, and zero, the symbol for nothingness, introduced significant simplifications into the use of numbers.

19. The law of inheritance of traits is significantly mathematical in nature.

20. The theory of correlation is very useful in suggesting causal relationships.

21. All numerical measurements are relative in value.

22. The distribution of errors conforms to a definite mathematical law.

From a historical point of view mathematics is vitally significant as an integral part of the culture of the world. Any educational program designed to meet the demands of the current educational philosophy which should omit mathematics from its content would be fundamentally incomplete. As a part of such a program mathematics should provide, out of its vast resources of relevant material, an abundance of stimulating opportunities for the development of those

> . . . powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.[1]

The major instructional emphasis should be directed systematically toward the *process* of thinking rather than toward the *product* of thinking. There should be conscious effort on the part of the teacher to cultivate in the mind of every student the proper evaluation of mathematics as an effective mode of thinking as well as an understanding of

[1] The National Committee on Mathematical Requirements, "The Reorganization of Mathematics in Secondary Education" (Boston: Houghton Mifflin Company, 1923), pp. 13–14.

its tool value in the practices of modern business and industry. The Chinese word for "crisis" is made up of two words, one meaning *danger* and the other meaning *opportunity*.[1] The crisis in secondary mathematics may thus be analyzed into the *danger* that prevails in the laissez-faire policy of accepting subject matter just because it shapes itself logically and pedagogically into rather teachable units of instruction and the *opportunity* that lurks among the unexplored challenges of functional presentation and interpretation of mathematical truths and techniques.

Factual versus Functional Instruction. In the curricula of the traditional school, subject matter was organized according to rather rigid patterns of logical sequence. The child was subjected to the severe disciplines of such a pedantic program in accordance with the precepts of a scholastic philosophy that defined education in terms of learning and knowledge. Today the pendulum has swung to the other extreme, and under the current demands of a pragmatic or naturalistic philosophy of education we find a tendency to have the child largely replace subject matter as the sole dictator of educational procedure.

A sound philosophy of education must chart the course of curriculum construction into the channel of significant educational procedure that lies between the whirling Charybdis of generalized social concepts on the one side and the formidable Scylla of mere factual information on the other. Taba has sounded a significant note of warning:

> Neither generalities too abstract and too remote to have any important significance in actuality, nor concrete and specific objectives devoid of any unifying generalizations and principles, can serve as an adequate basis for the guidance of education. Education, in order to be a constructive factor in human experience, has to be borne by concrete and actual situations, in which the specific can be seen in the light of general principles, in which the details and concrete elements are organized by means of concepts and logical relations. And, consequently, its aims are to be formulated so that they integrate the concrete actuality with the general principles, guiding ideas, and concepts.[2]

The spontaneous child-centered activity of the "new school" constitutes a basis for the selection and organization of educational objectives which is psychologically as unsound as the formally departmentalized factual information of the traditional school. Indeed, a

[1] "A Comprehensive English-Chinese Dictionary" (Shanghai, China: The Commercial Press, Ltd., 1937), pp. 522, 555, 1808.

[2] Hilda Taba, "The Dynamics of Education" (New York: Harcourt, Brace and Company, Inc., 1932), pp. 209–210.

society-centered program of secondary education is to be preferred to either a child-centered or a curriculum-centered program.[1] The child should be instructed in the techniques and information fundamental to efficient living before he is given "the liberty to swim in the stream of life." As the many obstacles are encountered in the fast-flowing current of the stream of life, it would be far better that the child be prepared to meet them in the calmness of competent preparation, fortified by those generalized disciplines which alone provide the means of self-adjustment, rather than in frantic despair to be engulfed in a whirlpool of haphazard learning. The educator should keep well in mind that the reconstruction of experience is essentially *factual* as well as *functional.* It thus becomes more or less axiomatic in the foundation of any instructional program that the more complex and advanced should be preceded by the more fundamental and elementary. This does not imply compartmentalization of subject matter but merely prescribes that instruction should be continuous with the development of the individual, factually as well as functionally. In the words of Demiashkevich:

> . . . at each level of education above the kindergarten, and in each type of school, the general educational method should be permeated with an effort on the part of the school to bring the educand to master an appropriate sequential curriculum. This effort seems to be demanded by the responsibility of the school toward the individual and toward the national commonwealth, as well as by indubitable lessons from the history of civilization regarding the achievements and failures of humanity. Spontaneous activity of the educand as a method of education, it appears, should enter the educative process only in so far as such activity is in harmony with and is conducive to the sequential study of an appropriate curriculum aiming at the development of desirable attitudes as well as skills and information.[2]

A Fundamental Curriculum Problem. What does such a philosophy of curriculum construction and instructional procedure imply as to the part mathematics should play in the formulation of a functional educational program? Number, the language of scientific effort and commercial activity; algebra, the syntax of functional dependence and abstract generalization; geometry, the grammar of systematic thinking, structural plan, and natural design, unite in the *façon de parler* of analytic composition to produce a rich and vital literature of inter-

[1] S. Everett, "A Challenge to Secondary Education" (New York: Appleton-Century-Crofts, Inc., 1935), pp. 21–26, 235.

[2] M. Demiashkevich, "An Introduction to the Philosophy of Education" (New York: American Book Company, 1935), pp. 121-122.

pretation, control, and progress. **The mathematical content of the curriculum should be so selected, organized, and presented that it will reveal to the child in the schoolroom and the layman on the street something of the essential significance of this language of algorism and abstraction as an interpreter of his immediate environment and the universe about him.**

Some acquaintance with numbers and skill in the fundamental operations of addition, subtraction, multiplication, and division is an educational objective to be taken for granted. The skills to be taught in this field and the types of problems to which these skills are applied should be determined by the kinds of arithmetical calculations which the ordinary American citizen has occasion to make. Elaborate and helpful investigations have been made to bring these fundamental operations into a position of prominence and recent revision of the curriculum in many school systems has resulted in great improvement in arithmetic instruction. In addition to skill in mathematics there needs to be developed an appreciation of the cultural value of mathematics, and of its usefulness as a mode of thinking and as a means of interpreting world affairs.

Closely associated with the fundamental arithmetical operations are the elements of intuitional geometry and applied algebra. Intensive technical study of more advanced mathematics should be offered to those whose vocational outlook, future education, or other special interests will make it necessary or helpful for them to use such knowledge.

New aspects of applied mathematics are constantly developing and the educational experiences of children and adults need to be extended to include them. For example, the presentation of numbers in graphic and tabular form is becoming extremely common. Children should learn the rudiments of graphic representation, particularly since this form of presenting data is at once so effective and so easy to misinterpret. The presentation of numerical data in graphic form is becoming a language with its own grammar and syntax. It is, however, a language which can ensnare and deceive the unwary. If children are to be taught that in the number 376, the 3 is in the hundred's column, the 7 in the ten's, and the 6 in the unit's, why should they not learn also the proper form for a chart and know that a chart which lacks certain features is potentially or actually dishonest and unreliable?

The ability to deal with number and form, the fundamentals of mathematics, has always been a basic human need. In an age such as ours where almost every phase of life is strongly marked by applied science and technology, the appreciation and use of basic mathematical skills and concepts offer significant assistance for self-realization.[1]

[1] W. G. Carr, "The Purposes of Education in American Democracy" (Washington: Educational Policies Commission, National Education Association, 1938), p. 3.

In the above words the Educational Policies Commission has outlined what might be considered a basic program in mathematics of value to everyone and has indicated the responsibility of schools to make provision for the further pursuance of special mathematical interests by those pupils who desire to do so. This basic program is the obligation of both the elementary school and the secondary school. One of the principal functions of the secondary school is to provide curricula for special interests, yet the elementary school should be held responsible for the provision of that basic preliminary training essential to the satisfactory quest of such interests. From the point of view of mathematics, a fundamental curriculum problem is thus seen to be the construction of a functional program of mathematical instruction that provides that mathematical content which is well oriented in the continuity of mathematical sequence and yet is functionally integrated with social experience. Furthermore, such a functional program must not only provide for "the realization of the democratic ideal" but also for the enrichment of the intellectual and cultural life of the individual.

Public education has by no means discharged its responsibility to the individual when it has *merely* provided opportunity for the acquiring of those technical skills and concrete concepts that are found to be most valuable when interpreted in terms of immediate or direct usefulness. Fundamentally, education should be a process of emancipation of the individual, not only that he may function more efficiently as an integral part of a changing social order, but also that, as an individual, he may have opened to him new avenues of rich intellectual and emotional experience in which he can continue to find that cultural satisfaction without which life is so incomplete. The educative process should, therefore, provide those experiences which not only would aid the individual to adapt himself to his environment and furnish him with the technical preparation necessary for earning a living, but also would encourage him to strive to improve his environment; inspire him to think independently and constructively, to form careful and unprejudiced judgments, to act courageously and with initiative; and serve to widen his horizons of intellectual endeavor, cultural interests, and recreational pursuits.

Three Cycles of Mathematical Instruction. The philosopher Hegel, in his analysis of the development of the universe, made use of his "triad of dialectics" composed of *Thesis*, *Antithesis*, and *Synthesis*. In the analysis of educational procedure this Hegelian "dialectical process" may be replaced by the "triad of didactics," *Apprehension*, *Application*, *Abstraction*. The use of this "didactical process" to

analyze instruction in mathematics gives meaning to Whitehead's "Rhythm of Education"[1] as it brings into relief three cycles of mathematical instruction, *viz.*, *Preparation*, *Foundation*, and *Specialization*.

The Preparation Cycle. The Preparation Cycle of mathematical development has its beginning in the apprehensions of the preschool period. These apprehensions come to the child in the form of sensations and perceptions which, at first, are extremely vague and which develop through an experiential process that is very slow. In the sensation of sight, the contact sensations in the hand, and the various kinesthetic sensations are laid the foundations for the later perceptions of size, shape, position, number, rhythm, distance, time, and weight. The function of the educative process in this first period of the Preparation Cycle is the informal provision of significant experiences. The mathematical development that takes place shows a transition from complete inability to adapt oneself to concepts of size, shape, or position to a distinction between "large" and "small," "round" and "square," "star-shaped" and "triangle," "upside down" and "right side up," "left" and "right"; from a vague concept of "more than one" to a limited ability to count; from an ability to distinguish between day and night to a realization of periods of the day and a concept of "yesterday" and "tomorrow." This accumulation of information through contacts with a large number of specific situations which involve similar responses constitutes the learning process for the child. There soon comes a time when this learning process takes on new significance. For further growth and development the educational program must provide opportunities for the child to become experienced and proficient in the basic techniques and fundamental concepts that the history of the race has shown to be significant.

With a child, as with an adult, the extent to which reactions are governed correctly will depend largely upon the extent to which his previous experience has been rich, wide, and varied. Education has as one of its functions giving knowledge which will enable a child to perceive things in their correct relations and, therefore, to react to them in line with the best interests of both the individual and society.[2]

The Application Period of this first cycle of mathematical development is that period in which formal attention should be paid to the development of desirable skills, attitudes, and habits in the four funda-

[1] A. N. Whitehead, "The Aims of Education" (New York: The Macmillan Company, 1929), pp. 24–44.

[2] A. H. Arlitt, "Psychology of Infancy and Early Childhood" (McGraw-Hill Book Company, Inc., 1930), p. 224.

mental operations with integers, common and decimal fractions, and the elementary processes of percentage. This program of instruction should be enriched by paying attention to graphs, simple formulas, and the geometry of measurement and simple design. These elements of mathematical information constitute a part of our racial heritage that is of significance to all. It is desirable that the child in the school have opportunity to become proficient in their use that later he might be able to apply them in making intelligent adaptation to his social environment. To attain maximum efficiency, educative subject matter must parallel in its continuity the development of the individual's experience both in its qualitative content and in its level of complexity. A systematic development of essential techniques for subsequent abstractions to desirable life situations is a sounder psychological procedure than is the selection of educational objectives according to the dictates of the spontaneous interests of immature and highly imaginative children who do not carry the patterns for their development within themselves.

In general, this period of application should begin with the third grade and extend into the seventh. Experience and experiment have fairly well placed the mathematical content of these five different grade levels. The primary emphasis should be on the acquiring of techniques and basic information in order that later desirable abstractions should not be empty of factual content. To motivate effort and to provide opportunity for appropriate integration of experience, functional projects should be used, but they should be the means to an end rather than the end itself. These projects can be organized around interesting stories of the home, school, games, animals, fowl, country life, transportation, communication, flowers, trees, parties, stores, seasons of the year, history, geography, etc. Such projects lay an intelligent foundation in the atmosphere of familiar and interesting incidents for the later generalized interpretations of techniques and information in terms of life situations.

The adolescent is standing on the threshold of youth and adulthood. His interests are many and active; consequently opportunity should be provided for contact with a wide range of desirable activities These activities should be presented in accordance with the abilities of the adolescent. He will have passed through a period of mathematical development that should have equipped him with the basic techniques and information essential to efficient living. He is now prepared for the Abstraction Period, the last period of the Preparation Cycle. In this period of mathematical development the subject matter should be

presented in its relation to the individual's efficient functioning as a member of his social order. The organization should be in terms of a definite functional program. A few suggestions as to significant social abstractions of mathematical content, which might be used in a program designed to provide meaningful integration of subject content, experience, and environment, are as follows: Mathematics in Nature and Decorative Design; Measurement and Construction; Mathematics in the Home, in Banking, in Business; The Importance of Mathematics in Invention and Discovery; How Mathematics Can Help in the Conservation and Protection of Life and Property; The Function of Mathematics in the Relation of Producer to Consumer; Mathematics as an Aid to Transportation and Communication; Mathematics in the Better Use of Leisure Time; How Mathematics Helps the Government in Its Work; The Concept of Relationship as Seen in the Laws of Nature. There are many other such suggestions that might be made but these are of obvious significance.

In the period of abstraction of mathematical processes to social procedure we have the culmination of the Preparation Cycle of mathematical development. At the end of the three years of this period of abstractions the student who does not plan to go further in his study of mathematics should have that mathematical information, and its generalized interpretation, that would equip him for efficient social adaptation to his environment. Such a program must also provide the mathematical training essential to further effective study in mathematics.

The Foundation Cycle. The educative process should be characterized by a continuity of effort rather than interrupted by a discontinuity of unrelated interests. To preserve this desired continuity for the student who desires to pursue further his study of mathematics, there should be present among the abstractions of the last period of the Preparation Cycle significant apprehensions of new techniques and procedures of the next or Foundation Cycle of mathematical development. These apprehensions should take the form of becoming familiar with the language and techniques of algebra and geometry. In fact there are many such apprehensions that can take place throughout the Preparation Cycle. Drawing of lines and simple rectilinear figures, measurement of lengths and drawing to scale are geometrical techniques that can be presented in simple form at a very early age. Use of formulas and graphs are algebraic techniques that can be introduced fairly early. In the seventh and eighth grades there should be systematic study of intuitive geometry, further study of graphs and

formulas, and possibly toward the end of the eighth grade an introduction to simple equations and the simpler trigonometric ratios. The ninth grade should then very definitely develop the fundamentals of algebraic technique and acquaint the student with the techniques of numerical trigonometry. There are some who recommend that there should also be an exploratory unit of demonstrative geometry placed toward the end of the ninth grade.

This should end the Apprehension Period of the second or Foundation Cycle of mathematical development. This second cycle, which begins with the introduction to the techniques and processes of algebra and geometry and closes with a well-rounded course in differential and integral calculus, has been called the Foundation Cycle, as it is during this time that the individual comes into definite contact with that mathematical information which forms a foundation for significant development in mathematical thought.

Those apprehensions of subject matter and important techniques which take place in the work of the grades and the junior high school should prepare the student for earnest effort to acquire proficiency in those techniques and that information essential for further mathematical development. The Application Period of this second cycle offers the opportunity to secure this proficiency through systematic study of the fundamental techniques and basic information of algebra, trigonometry, and both synthetic and analytic geometry. The importance and power of the concept of functional dependence should be carefully emphasized and constantly displayed. Great stress should be placed upon the significance of the generalizations of algebra, such as the extension of the number system to include negative and complex numbers, the convenience and power of algebraic notation, and the conciseness of algebraic processes. Geometry should be studied as a vehicle for postulational thought as well as a science of measurement and a study of constructional techniques. The importance of geometry in industry, science, architecture, and nature should be constantly emphasized.

The concept of functional dependence stressed in the study of trigonometry and analytic geometry together with the limiting process, the concept of infinitesimals, and the techniques of differentiation and integration constitute a medium for the significant abstractions of the techniques of algebra and geometry. The generalizations of processes and information that take place in this Abstraction Period help to round out a program of mathematical education that should provide

an efficient mathematical foundation for general cultural information and for further specialized study. This specialization can take the direction of any one of several different channels, the more significant of which are the research worker in pure or applied mathematics and the teacher of secondary or college mathematics.

The Specialization Cycle. The generalized concepts and processes of the Foundation Cycle become the newly apprehended skills and abilities of the Specialization Cycle of mathematical growth. With these newly acquired elements of mathematical procedure the student is now ready to enter a period of intensive application and study to become efficient in their use in a chosen channel of specialized effort. The completion of the program of this Application Period is usually heralded by the acquiring of an advanced degree This, in general, ends the period of supervised study for the individual. He now enters upon the final Abstraction Period, in which he must make his own generalizations and abstractions. If he is the research worker, he must use the techniques and information in the extension of the frontiers of knowledge. If he is the teacher, he must know how to make of his advanced information a rich source of material to be used in the inspired instruction of new students journeying along the road of mathematical progress.

A Functional Program of Mathematical Instruction. Such a cycloramic view should enable the educator to secure a more significant educational perspective of the mathematical development of the individual. He should see the helical curve of mathematical development as it spirals its continuous path through those distinct yet overlapping periods of Apprehension, Application, and Abstraction in its upward course through the cycles of Preparation, Foundation, and Specialization. This curve starts its upward journey at birth and, having wound its course through that period of life in which preparation is made for significant life experience, next meets the line of physical growth at the passage from childhood to adolescence. At this point it will terminate for some, having given them mathematical training sufficient for successful adaptation to their social environment; for others the curve will pursue its continuous course as it proceeds to build a firm foundation for later specialized study. The point at which the curve next meets the line of physical growth is at the passage from youth to adulthood. This will terminate the mathematical development of other individuals at a time when they should have acquired an efficient mathematical background for general cultural information; for still

others the curve will lead the way into any of several significant channels of specialized effort and follow its continuous climb throughout the life of the individual.

Such a perspective of mathematical development emphasizes the essential necessity of providing significant mathematical content at appropriate stages of educational experience; the importance of timely abstractions; the expediency of intelligent integration; the futility of generalizations empty of factual content; and, finally, the pertinence of a proper balance between the factual and the functional in the instructional program.

Exercises

1. What implications for educational change are to be found in the changes which life in the United States has undergone in recent years?
2. What do you consider the major responsibility of the school to society?
3. What are some of the more significant proposals made for meeting this responsibility?
4. What are some of the more important criticisms of the subject-matter form of organization of the curriculum?
5. What are some of the more outstanding weaknesses of the large-unit, or activity, form of organization of the curriculum?
6. Discuss the following statement and its implications relative to the mathematics curriculum of the secondary school: "The reconstruction of experience is essentially factual as well as functional."
7. What do you consider the most significant trends in the organization of secondary mathematics?
8. What are some of the effects which the activity program has had on the mathematics curriculum of the secondary school?
9. In your opinion what place should mathematics occupy in the program of modern secondary education?
10. What should be the more significant characteristics of the modern program in secondary mathematics?
11. What implications are to be drawn from the modern theory of transfer of training as to the place for mathematics in the secondary curriculum?
12. Outline an argument which you might present in defense of mathematics as a significant part of a program in general education at the high-school or college level.
13. What would be the major aspects of the content of mathematics in such a program of general education?
14. What are some of the major instructional problems likely to arise in the presentation of such a program in mathematics?

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PART II

THE IMPROVEMENT AND EVALUATION OF INSTRUCTION IN SECONDARY MATHEMATICS

CHAPTER IV

MATERIALS OF INSTRUCTION: CURRICULAR CONSIDERATIONS

Any endeavor which is conceived intelligently will aim at certain outcomes or results. Only to the extent that it does this is it purposeful, and only to the extent that it is purposeful can it be consistently effective. Instruction in mathematics is no exception to this principle. If it is to be worth while, it must be planned with the idea of attaining certain objectives which represent those values thought to accrue to the study of the subject. The formulation of these objectives, therefore, should be the first step in the organization of the course of study.

The matter of determining the broad general objectives of mathematical instruction at the secondary level is approached logically through a consideration of the relation which mathematics bears to secondary education as a whole and through a consideration of the contributions which it may be expected to make toward the attainment of the general objectives of any phase of the secondary program. The broad objectives of mathematics at any grade level of instruction should be consistent with, and contributory to, the larger aims for all mathematical instruction Similarly the general aims of each unit of a given course should contribute toward the attainment of one or more of the broad aims of the course. It is thus possible to set up a rational basis for determining appropriate content of mathematical courses at the various grade levels.

The selection and organization of the subject matter of the mathematics courses and the planning of student learning activities are extremely important and exceedingly difficult undertakings. In recent years the complexity of the task has been increased vastly by accelerated scientific, technological, and social change; by the rapid growth of the secondary schools; by increased diversity of abilities and interests of the students; and by the already tremendous expansion of the curriculum. The war itself brought added complications, and the readjustment to a postwar era with its swiftly changing conditions is bringing still others.

Out of this welter of uncertainty and conflicting ideas, one fact

emerges clearly: tradition alone is no longer regarded as sufficient justification for the selection or the organization of the subject matter of any course. Courses of study are rightly coming to be thought of as means to ends rather than as ends in themselves, and subject matter can be justified only insofar as it gives promise of being the means through which certain desirable educational objectives can be attained. If the objectives are clearly formulated, they provide a basis for selecting and arranging the subject matter in such a way as to make probable its most effective contribution toward the attainment of the desired goals. The selection of objectives should come first: otherwise a planless, wasteful, and ineffective course is likely to result.

Classification of Objectives. The broad general objectives for mathematical instruction have been variously classified. Among the earlier authoritative works to give particular attention to this problem, the classic Report of the National Committee on Mathematical Requirements stands pre-eminent in the matter of clarifying these aims, just as it was for at least two decades the dominant influence in shaping the more forward-looking courses of study. Its classification of the broad aims of mathematical instruction as practical, disciplinary, and cultural has been mentioned in a previous chapter. This classification has been extremely useful as a guide to teachers, supervisory officers, curriculum makers, and textbook writers. Indeed it has exerted so much influence that it has come to represent a sort of standard frame of reference for objectives in this field.[1]

Other classifications have been proposed, however, which are sufficiently different in detail to make them worth noting at this time. Young,[2] for example, discusses the principal values of the study of mathematics under three general headings, (1) practical values of mathematics, (2) mathematics as a mode of thought, and (3) other functions of mathematics. Under this third and rather indefinite heading he mentions values which are in the nature of attitudes, habits, and ideals.

Breslich[3] classifies the principal aims as (1) understandings, (2) skills, (3) problems and methods, (4) appreciations, (5) attitudes, and (6) habits.

[1] The National Committee on Mathematical Requirements, "The Reorganization of Mathematics in Secondary Education" (Boston: Houghton Mifflin Company, 1923), pp. 6–13.

[2] J. W. A. Young, "The Teaching of Mathematics" (New York: Longmans, Green & Co., Inc., 1924), pp. 13, 17, 41.

[3] E. R. Breslich, "The Technique of Teaching Secondary-school Mathematics" (Chicago: University of Chicago Press, 1930), pp. 190–191.

Blackhurst[1] lists them as (1) attitudes, (2) concepts, and (3) information.

Schorling[2] presents the general objectives for junior-high-school mathematics under four headings, (1) attitudes, (2) concepts, (3) abilities, and (4) information. In another and later book[3] this same author gives a more detailed and more definitive list of 11 general objectives.

Smith and Reeve,[4] in a lengthy discussion of the broad objectives of junior-high-school mathematics, distinguish between mathematical objectives and general objectives, listing and discussing certain ones under each of these categories. In a subsequent discussion these same authors consider in detail a large number of more specific objectives.

Minnick[5] discusses the general aims of mathematical instruction under the four headings of (1) practical values, (2) preparatory values, (3) cultural values, and (4) disciplinary values.

In the Report of the Joint Commission[6] there is given a statement of the general objectives of secondary education. The Commission does not set forth a categorical list of aims for mathematics, but its discussion of the general objectives for mathematical instruction consists of its interpretation of the contributions which mathematics can make to the attainment of these broad objectives of secondary education.

In 1945 the Second Report of the Commission on Post-War Plans[7] was published. This comprehensive report deals with proposals for improving mathematical instruction throughout the entire school system, but the emphasis is mainly on the program for the secondary

[1] J. Herbert Blackhurst, "Principles and Methods of Junior High School Mathematics" (New York: Appleton-Century-Crofts, Inc., 1928), p. 15.

[2] Raleigh Schorling, "A Tentative List of Objectives in the Teaching of Junior High School Mathematics" (Ann Arbor, Mich.: George Wahr, 1925), pp. 97–116.

[3] Raleigh Schorling, "The Teaching of Mathematics" (Ann Arbor, Mich.: Ann Arbor Press, 1936), pp. 24–28.

[4] D. E. Smith and W. D. Reeve, "The Teaching of Junior High School Mathematics" (Boston: Ginn & Company, 1927), pp. 22–35.

[5] J. H. Minnick, "Teaching Mathematics in the Secondary Schools" (New York: Prentice-Hall, Inc., 1939), pp. 38–4[illegible].

[6] Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, *Fifteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1940), pp. 21–31.

[7] Commission on Post-War Plans, Second Report, *The Mathematics Teacher*, **38** (1945), 195–221.

school. While the report is concerned mainly with a *program* of improvement, which has already been mentioned, and contains no assembled categorical list of general objectives, such objectives are either stated or implied at various places in the report. Perhaps they can be broadly summarized as follows: (1) The main objective of the sequential courses in the ninth grade and in the senior high school should be to develop mathematical power. (2) The main objective of the work in the seventh and eighth grades and in the general mathematics courses in subsequent grades should be to develop functional competence in mathematics. This would be a helpful distinction in any case, and it has been needed for a long time. Its appearance in this report is particularly helpful since the Commission has had the wisdom and courage to define this "functional competence" in terms of a Check List of 28[1] specific mathematical attainments. These items in fact constitute the statement of 28 *specific* objectives for the mathematical instruction of *all* students.

Other classifications and discussions of objectives exist, but those which have been given here are sufficient to show the general tendencies of writers in considering the values and outcomes of mathematical instruction in the secondary schools. With somewhat varying emphases and degrees of detail they seem to cover, almost without exception, much the same ground as that indicated in the Report of the National Committee on Mathematical Requirements. Attitudes, understandings, ideals, appreciations, and the like, are definitely cultural traits whose acquisition involves disciplinary and cultural values. Skills, abilities, and information are often practical as well as cultural. The differences in statement represent little more than differences in emphasis on various aspects of the practical, disciplinary, and cultural objectives.

As to which, if any, of these categories should predominate, no overall single statement will serve. The abilities and the plans of the students and the types of courses and the grade levels under consideration would have important bearings on this question. In 1933 an extensive sampling study was reported[2] which indicated that at that time the practical objectives were actually receiving a good deal more emphasis than either the disciplinary or the cultural aims in junior-

[1] This list was increased to 29 items in the Commission's subsequent Guidance Report, *The Mathematics Teacher*, **40** (1947), 315–339.

[2] Edwin S Lide, Instruction in Mathematics, *Bulletin* 17, Office of Education, 1932, National Survey of Secondary Education, *Monograph* 23 (Washington: Government Printing Office, 1933).

high-school mathematics, while in the senior-high-school courses the practical aims received less emphasis than before and the other objectives all increased in prominence. We shall see that theoretical considerations suggest that this is not only to be expected but that it is logically desirable. Since no comparable study has been reported in the intervening years, it is not possible to say with entire certainty whether the same allocation of emphasis holds today, but empirical observation gives some basis for believing that no great change in relative prominence of these broad objectives has taken place up to this time.

Mathematical Values and Mathematical Needs. It seems reasonable to postulate that the educational values of mathematics and the mathematical needs of people should constitute the principal basis for determining the objectives of instruction, but there has been, and there still is, disagreement as to what these mathematical needs include. On the one hand, there are those who feel that, so far as general education is concerned, these needs consist of the narrowly practical or tool functions of the subject. Other leaders, however, contend that this is but one of the values that may be derived from the study of mathematics and that the educational needs of people include other less immediate, less tangible, less specific, and more general values which are thought to be definitely associated with mathematical education.

> . . . the word *need* will be understood to denote not only such knowledge or capacities as may be indispensable, but also attainments that may profitably be used in either a utilitarian or a cultural manner. In a very real sense such knowledge and capacities are actual needs, to be provided for by the schools . . . .[1]

Under this broader interpretation it is evident that the mathematical needs of people embrace a range of values much more inclusive than the purely instrumental applications of the subject. The Joint Commission, in its analysis of mathematical needs,[2] makes this clear through the enunciation of a wide variety of situations with which mathematics has significant points of contact. This analysis specifies clearly the practical values of the subject, but at the same time it places appropriate emphasis upon the cultural and preparatory values.

Practical Values of Mathematics. The practical values of mathematics fall into two more or less distinct categories: utilitarian and preparatory. From the standpoint of immediate utilitarian values

[1] Joint Commission, *op. cit.*, p. 207.

[2] *Ibid.*, pp. 207–216.

there are certain phases or parts of mathematics which are indispensable tools in the intellectual equipment of the intelligent citizen. These are of universal importance, and their acquisition by every child should be regarded as essential. As illustrations of such distinctly practical values of mathematics we might list the operations of arithmetic, the concepts of measurement, the meaning and interpretation of graphic representation, the simpler notions of statistics, the algebra of the formula and the linear equation, the effect of the use of approximate data, familiarity with the more common geometric forms, knowledge of the mensuration of these forms, the understanding of how mathematical concepts and processes are applied to problems of everyday experience, and the mastery of a wide variety of mathematical concepts and terms as a basis for intelligent reading of contemporary articles in current periodicals. Such attainments as these should constitute a very important part of the educational equipment of every citizen. Evidence of the validity of this proposal under wartime conditions can be found in abundance in certain wartime educational reports of the armed services[1] as well as those of various nonmilitary agencies. With respect to postwar conditions this position is perhaps most pointedly emphasized in the Check List on "functional competence in mathematics" prepared by the Commission on Post-War Plans, and to which previous reference has been made already.[2]

The items to which reference has been made above represent "practical" values in a very direct sense, and it is not surprising to note that they have been drawn almost entirely from the field of junior-high-school mathematics. In the conventional senior-high-school courses the utilitarian values of demonstrative geometry and of the more theoretical aspects of algebra are usually less evident, except for students who expect to continue still further in mathematics or to engage in work or studies for which algebra and geometry would be important as background. It is true that such courses as elementary statistics and "consumer mathematics" at a slightly advanced level are beginning to find a place in the senior-high-school program, and these courses do have very practical bearings.[3] But such courses are still exceptional and marginal, rather than typical and central, parts of

[1] See, for example, the report, Essential Mathematics for Minimum Army Needs, *The Mathematics Teacher*, **36** (1943,) 243–282.

[2] *Cf.* pp. 45–46.

[3] Commission on Post-War Plans, *The Rôle of Mathematics in Consumer Education* (Washington: The Consumer Education Study, National Association of Secondary-School Principals, 1945).

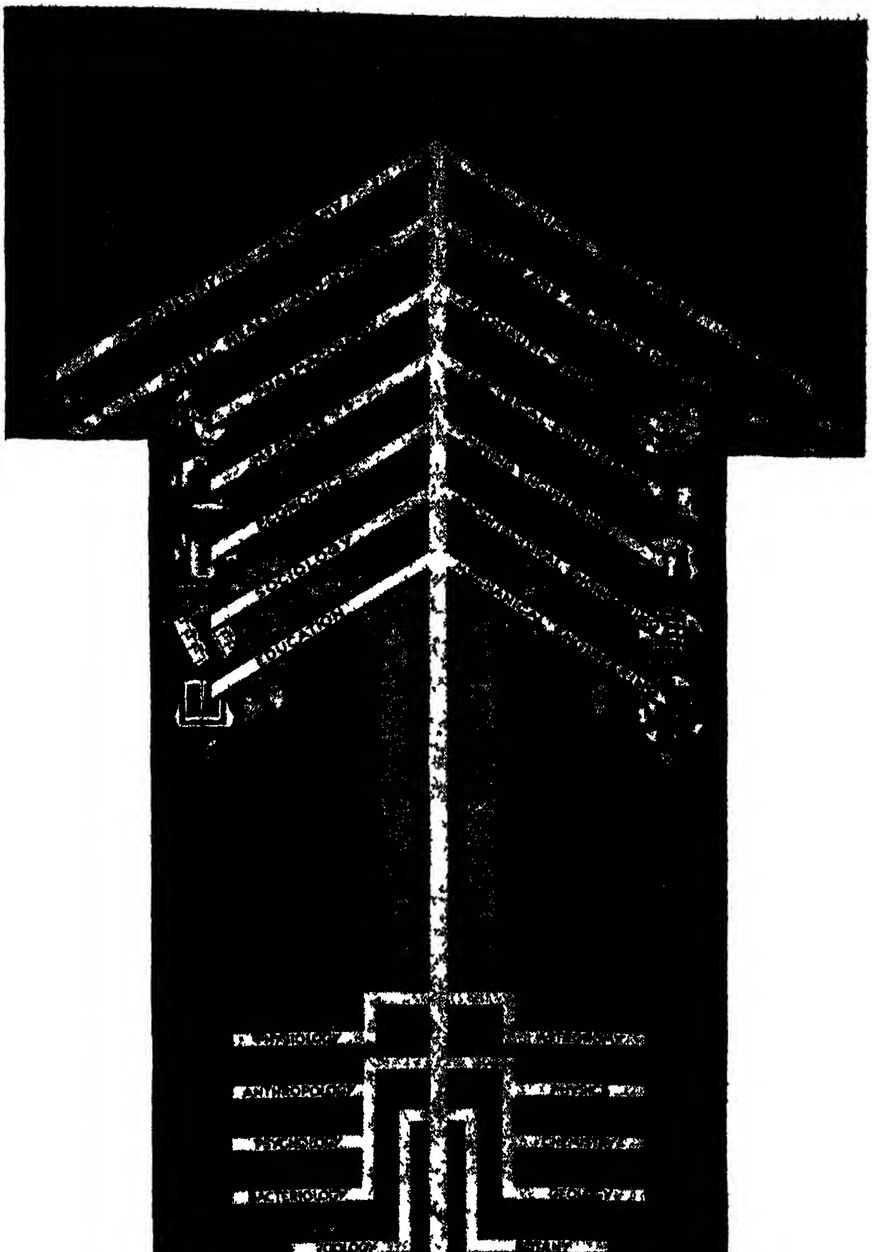

FIG. 1. A century of progress mural by John Norton, 1933. (*Copyright Museum of Science and Industry, Chicago; reproduced by special permission of Museum of Science and Industry, Chicago.*)

the senior-high-school program. It follows, then, that on the whole the objective of universal and immediate practical value is one which is very pronounced in the mathematics of the junior high school but which diminishes perceptibly with respect to the later conventional courses.

There is still another group of practical values of mathematics which should be taken into account although they are of significance in a much more restricted sense than those already mentioned. It should not be overlooked that increasing numbers of students will want to pursue advanced studies in the later years of high school or in college. Careful analysis will reveal that for a good deal of this work mathematical training beyond the minimum indicated above is an increasingly important asset. In a number of fields it is not only an asset but a definite necessity, especially at the collegiate and professional levels. The Joint Commission,[1] in its analysis of mathematical needs, has indicated that advanced mathematics finds frequent application, not only in the obvious cases of higher mathematics, engineering, and the physical sciences, but also in various specialized branches of the earth sciences, biology, agriculture, the social studies, commerce and industry, psychology and education, and even in philosophy and esthetics. A requirement of one to two years of college mathematics is no longer uncommon for students majoring in these fields. In recent years, especially since the war, many students who have entered college without the requisite background of high-school mathematics have found themselves barred from entering courses which they wanted. Furthermore, there have been occasions when such students have had their work, and even their graduation, delayed by the need to make up such deficiencies. In view of this situation it seems clear that, for all students who will enroll in any college courses requiring a preliminary background of high-school mathematics, the propaedeutic value of such a background will become a very practical consideration indeed.[2]

General Values of Mathematics. In this era of curriculum revision there is, in some quarters, a disposition to discount the worth of mathematical courses at the upper levels of the secondary school for the reason that the emphasis shifts there in the direction of propaedeutic and general values with a consequent reduction of emphasis upon immediate utilitarian values. This seems an unfortunate view to take

[1] Joint Commission, *op. cit.*, pp. 211–216.

[2] See Judson W. Foust, The Responsibility of the Mathematics Teacher in Curriculum Building, *The Mathematics Teacher*, **36** (1943), 104; see also the Guidance Report of the Commission on Post-War Plans, *op. cit.*

because, in effect, it denies the reality, or at least the importance, of what are often called the intangibles: appreciations, disciplines, attitudes, ideals, etc. Moreover, it fails to recognize that the well-ordered human mind is so constituted that some of these intangibles, as well as a certain amount of sheer intellectual play, are conducive and perhaps necessary to its well-being. These things should be taken into account in considering the objectives of mathematical training in the schools, because they constitute a group of values which are very real, especially at the upper levels of the secondary school, even though they are not associated in the minds of all people with this subject. Whether definite provision is made for them in the school training of young people or not, it is inevitable that they will be acquired in one form or another in the course of the social, intellectual, and emotional experiences which are inescapable in the process of growing up.

Any discussion of the general values of mathematical instruction must take into account the question of the transfer of training. That the implications of transfer are of extreme and far-reaching importance may be judged by the vigorous controversy which has centered about it for several decades. It is doubtful whether any other question of educational theory has ever had a more profound effect upon thought and practice in the matter of curriculum construction than has the controversy over mental discipline and transfer of training. It has been pointed out in an earlier chapter that the theory of mental discipline occupied a prominent place in the educational philosophy of the early American secondary schools, but later the advent of a mechanistic psychology cast grave doubt upon its validity. Furthermore, psychological experimentation has demonstrated rather conclusively that transfer is not complete, automatic, and inevitable. Many uncritical people have interpreted this to mean that transfer is nonexistent, although no experimental evidence has ever warranted this conclusion.

These contrasting points of view carry very evident implications concerning the place of mathematics in the educational program of the secondary school. Acceptance of the theory of transfer would imply that mathematics should be accorded a place of high importance in the program, regardless of any practical values that it might have, whereas denial of the principle of transfer would imply a consequent denial of any important outcomes other than those having distinctly practical or purely cultural bearings. In other words, the one view would encourage the study of mathematics for its disciplinary values throughout the entire secondary-school program, while the other view would sanction only those courses which can be justified from the standpoint

of their direct social and practical values. The one view would thus endorse a broad comprehensive program of mathematical instruction; the other, a definitely limited offering in this field.

There can be no doubt that many extravagant and unjustified claims have been made with regard to the disciplinary values of mathematics. On the other hand, the willingness to accept uncritically the "no transfer" dictum has undoubtedly led to statements and beliefs equally extravagant and quite as far from the truth. Competent psychologists today are agreed that the truth lies between the two extreme positions. There is no longer any doubt that transfer does take place.[1] The real question is as to the manner in which it takes place and the circumstances that are most favorable to its consummation. The most competent evidence indicates that, in order to achieve the disciplinary values of any subject, it is necessary to teach that subject with that specific purpose in view.

In a broad sense the disciplinary effect of sound mathematical study may be thought of as involving potentially such values as the awareness of, and insistence upon, precision; the establishment of self-reliance and the self-imposition of responsibility for information, procedure, and results; persistence in the face of difficulty; habitual insistence upon the precise use of language and upon clarity and precision in definition and statement; the ability to discriminate between a mere assertion and an inference; the habitual testing of inferences for consistency with known or given conditions; the ability to discriminate between sound and specious argument and between valid inferences and unwarranted inferences; awareness of the nature of postulational reasoning, of the arbitrary nature of hypothesis and definition, and of the inevitable but contingent nature of conclusion; the ability to generalize relationships and to apply generalizations; the ability to build a consistent argument; and the ability to eliminate emotional or prejudicial factors from an argument. In particular, the ability to generalize meanings, symbols, relationships, and processes and to apply such generalizations to new situations represents transfer of the most genuine and vital sort. In fact, this is precisely what is implied by the expression "func-

[1] In this connection, see, among others, the following references:

W. C. Bagley, "Education and Emergent Man" (New York: Thomas Nelson & Sons, 1934), pp. 82–93.

Charles H. Judd, "Education as Cultivation of the Higher Mental Processes" (New York: The Macmillan Company, 1936), pp. 198-201.

William Betz, The Teaching and Learning Processes in Mathematics, *The Mathematics Teacher*, **42** (1949), 49–55.

tional" mathematics, which has come into use as the dominant idea for the courses in general mathematics as well as for the more formal sequential courses. Indeed, this aspect of transfer would seem to lie at the very root of all really functional education. It is implied in every application and every interpretation of any concept or circumstance, for correct interpretation must form the basis of any intelligent application, whether to a problem in physics or geometry or to a business or social situation. A denial of this sort of transfer value seems utterly inconsistent with the advocacy of teaching for meanings and of making education really functional.

Complete attainment of all these values in every case is neither claimed nor expected, but no reasonable person would question the desirability of these attributes, and since it appears that proper instruction in mathematics can be expected to contribute to their attainment, they may be set down very properly as important objectives for mathematical education.

Current Issues. The foregoing sections of this chapter provide a background against which we can consider some of the practical questions that have to be faced in laying out a program in mathematics and building the courses at the various grade levels. These questions will be centered, of course, around the focal consideration of what will serve best the real needs of the students. It has been recognized for a long time that students at any grade level exhibit wide differences in mathematical aptitude, in academic and vocational interests, and in their plans and expectations for the post-high-school years. It therefore follows that the real mathematical needs of some students will differ from those of others. It is also clear that the ultimate objectives of the sequential courses in the senior high school are not identical with the aims of the courses in general mathematics. The big issues, then, refer to the over-all question of what work should be offered at what grade levels and who should take it.

This dual question is so broad that it needs to be broken down into a good many subquestions. Indeed, one could hardly hope to set down all of the implied questions, but the following list may serve to indicate some which appear to be of major concern.

1. Is there a common minimum core of mathematical attainment which should be required of all students alike?

2. Should the work in the seventh and eighth grades be differentiated, or should a single unified program be required of all the students in these grades?

3. Should all ninth-grade students be required to take either algebra or general mathematics? If so, who should take which?

4. What should be included in ninth-grade general mathematics?

5. What students in the senior high school should take the sequential courses in algebra, geometry, and trigonometry?

6. What can be done to improve the sequential courses so that they will serve more effectively in developing genuine mathematical power?

7. Should special, or "second-track," courses be developed for the senior high school, to serve the general educational needs of those students who lack the ability or the interest to profit substantially from the sequential courses? If so, how extensive should these courses be, and what should they include?

8. Should students who do not take the sequential courses in mathematics be required to take at least one year of general mathematics in the senior high school?

9. How could a "double-track" program be offered in a small high school?

10. How can general mathematics be made to have in the eyes of students and parents (and also of teachers) a status of "respectability" equal to that commonly accorded to the sequential courses?

11. If it can be assumed that the secondary school should aim to develop mathematical literacy in all its students, of what, precisely, should this mathematical literacy consist?

12. What place should arithmetic have in the secondary-school program, and what should be done to make it fulfill its function more effectively than it has in the past?

13. Should the attainment of some specified minimum standard of proficiency in arithmetic be required as a condition for graduation from high school?

14. Commercial arithmetic is often given in high school. Should a special course in shop mathematics be offered too?

15. Should a course in consumer mathematics be given in the twelfth grade? If so, what should it include? Should it be a required course for all seniors? If not, who should take it?

16. Should more emphasis be placed on teaching for meaning? If so, should the increased emphasis be on social meaning (applications) or on mathematical meaning (concepts and relationships)?

17. If such outcomes as appreciation, generalization, critical judgment, etc., are important, better methods are needed for evaluating attainment in these directions. What methods could be used effectively to this end?

18. What can be done to challenge the interests and serve more effectively the needs of the very superior students?

19. What provision should be made in the junior college for students whose needs are not well served by the regular sequential courses in college mathematics?

Proposals for a Modern Program in Mathematics. As has been said, the issues in question with respect to the curriculum and the con-

cern of interested people naturally center about two main and interdependent questions, (1) what kinds of courses in mathematics should be provided in the secondary schools, and (2) who should take these courses? Curriculum and guidance committees in many schools have studied these questions and their implications and have evolved a variety of curricula and literally thousands of course outlines. For a long time there was no recognized agency to serve for clearing and coordinating these many plans or for proposing policies on more than a local basis, but the growing awareness of the need for broader coordination of policies led to the establishment of such an agency early in 1944. As indicated in Chap. II,[1] the National Council of Teachers of Mathematics, looking ahead to the postwar period, created in that year a commission known as the Commission on Post-War Plans. This important commission was charged with the responsibility of studying the issues and of proposing policies and plans for the mathematics program for the consideration of all the schools in the country. The Commission subsequently published four reports, the first two of which deal specifically with curricular proposals. It is believed that the consensus of competent opinion about current curricular issues can best be brought into focus by noting the recommendations carried in these reports. These proposals for a modern program in mathematics have been summarized on pages 45 to 47. Readers will find it helpful at this point to review these proposals and the supporting arguments which are set forth by the Commission. They are of extreme importance. Their weight is already beginning to be felt, and they are probably destined to play a large part in shaping the mathematics curriculum for the foreseeable future.

Need for Improving the Courses. The selection and arrangement of the actual subject matter in the different courses has received a steadily increasing amount of attention since the beginning of this century. It has been felt for a long time that the traditional courses in arithmetic, algebra, and geometry have not been altogether satisfactory, and more recently there has been a growing conviction that the courses in the junior college are also in need of revision if they are to serve all the students. It was this feeling of dissatisfaction that gave rise to the reorganization movement which has gained momentum through the years. It has produced notable beneficial changes, especially in the junior-high-school courses It has also brought improvements into the sequential high-school courses, though much remains to be done there.

[1] *Cf.* pp. 44–48.

It has become increasingly apparent, however, that there is a large fraction of the senior-high-school and junior-college group to whose mathematical interests the sequential courses do not seem to be the answer, and the weight of opinion indicates that courses of a different kind ought to be developed for such students. Mention has already been made of proposals for a double-track program in mathematics in the senior high school and the junior college. Such a program could bring about changes and benefits of great significance at these levels, provided that suitable and really significant courses can be worked out for the "other track." Perhaps no other curriculum problem calls more urgently for solution at this time than does the development of such courses.

Programs and courses can never become perfect, and even if they could, they could not remain very long in static perfection. The need of searching for improvements is an endless one, and the vigorous prosecution of this search is the mark of an alert and healthy profession. Never before have so many teachers been engaged so seriously in the attempt to work out improvements in the mathematics of the secondary school. It is significant that almost eleven hundred articles dealing with one phase or another of mathematical education have been published in the United States in the past ten years. It is significant also that the college and university teachers, who so often have held aloof, are beginning to make some very helpful contributions. It is essential that all the values of the present courses be conserved. But it is also essential that complacency be avoided. If improvements are to be made, it will be necessary to couple imagination and open-mindedness with sound appraisal and evaluation of outcomes in a vigorous effort to do our job better than it has been done in the past.

Mathematics in the Seventh and Eighth Grades. The recommendations of the National Committee on Mathematical Requirements for the content of the courses in the seventh and eighth grades have been accepted so generally that in effect they have come to form the basis of these courses. On this matter the subsequent major report of the Joint Commission and the Second Report of the Commission on Post-War Plans have both been substantially in harmony with this earlier report. It is true that the recommendations of the Joint Commission go into more detail respecting the grade placement of subject matter and that those of the Commission on Post-War Plans help considerably in clarifying and blueprinting certain long-range considerations which should govern the organization of the detailed courses. But there is no disagreement among these three major reports. Each has focused

on a particular major facet of the problem, and this supplementation has been progressively helpful. These three reports undoubtedly have been very influential in shaping the content of the work for the seventh and eighth grades into a unified and functional block. There is almost universal agreement that this is not the place to begin differentiated courses. The mathematical program for these grades should be essentially the same for all normal pupils.[1]

For the most part, general mathematics has replaced arithmetic in these grades. Authors of textbooks differ with respect to the order and arrangement of the material, but they agree pretty generally on what is to be included. This material is drawn mainly from the fields of arithmetic, informal geometry, graphic representation, the beginnings of trigonometric work, and elementary algebra. It has been the aim of the authors of most recent textbooks for the seventh and eighth grades to weave together elements from all these fields, especially arithmetic, informal geometry, graphic work, and the use of simple formulas, in such a way that they will concurrently enrich and motivate each other.

Particular emphasis is placed upon a continuation and extension of arithmetical work and upon informal geometry as parallel central themes running throughout the two-year block. Running concurrently with these central themes we find supplementary experiences drawn from graphics, trigonometry, and the history of mathematics, related activities in various fields, and mathematical recreations. Emphasis is being placed upon social implications and upon the processes that are socially useful. Much effort is being made to improve the program of drill and maintenance and the problem material, and increasing attention is given to laboratory techniques and the use of multisensory aids.[2] Such a program satisfies the principles of continuity, flexibility, and cumulation laid down by the Joint Commission as basic considerations, and it provides appropriate centers of emphasis about which the systematic work of each grade can be organized. With good teaching it can go far toward developing the "functional competence" in mathematics demanded by the Commission on Post-War Plans.

Mathematics in the Ninth Grade. The changing character of the secondary-school population has accentuated the curricular problems

[1] See the Second Report of the Commission on Post-War Plans, *op. cit.*, pp. 203–205.

[2] See Raleigh Schorling, Trends in Junior High School Mathematics, *The Mathematics Teacher*, **35** (1942), 339–342.

arising from the diverse abilities, interests, and real needs of the students. In the seventh and eighth grades the over-all objective is the same for all the students, viz., the development of general mathematical literacy. Therefore in these grades a common course is indicated in spite of differences among the students. With the ninth grade, however, some of the students will wish to, and should, begin work on the series of sequential courses. There will be other students who, though still needing more mathematics, will have little interest in systematic algebra, inadequate ability to profit much from it, and probably little eventual need for it. These students need a different kind of mathematics. It is at this point, then, that differentiated courses, or a double-track program, should first appear. The position of the Commission on Post-War Plans[1] is explicit on this, especially as regards the larger high schools where staff and facilities are at least reasonably adequate. The disadvantages of offering only a single course can be seen clearly, and they are serious. If only algebra is offered it is likely to be a diluted algebra. When this happens, standards of achievement are inevitably lowered and even then frustration and failure are apt to run high. On the other hand, if only general mathematics is given, there is inevitable waste of time and probable deterioration of interest among the better students, and delay for those who ought to begin their work on the program of sequential courses at this time. It does not seem possible for any single course to meet the needs of both types of students.

On the assumption, then, that a double-track program should be offered in the ninth grade, it remains to specify the nature of each course. Here again the Commission on Post-War Plans provides help by giving a definitive description of general mathematics for the ninth grade.

General mathematics for the ninth grade is here defined as a course that includes and emphasizes the elements of functional competence as outlined in the Check List on pages 197–198 of this (second) report. It has been suggested earlier that the task of insuring functional competence cannot be completed for all pupils in the first eight grades. For many this task must be continued at least through grade 9. The main purpose then of a general mathematics course in the ninth grade is to provide such experiences as will insure growth in understanding of the basic concepts and improvement in the necessary skills. . . . [2]

[1] Second Report, *op cit*, pp 205-213

[2] *Ibid.*, p 206 Also see Raleigh Schorling, Mathematics in General Education, *School Science and Mathematics*, **49** (1949), 298

Furthermore, the Commission gives a rather long and detailed description of what it considers good practice in the teaching of first-year algebra.[1] Algebra teachers will find in this list many excellent and helpful suggestions pertaining both to the content and methodology of instruction.

The advantages of the double-track plan apparently are coming to be recognized by a good many school administrators. Evidence from an extensive sampling inquiry in 1947 indicates that in approximately half of the secondary schools which enroll more than 100 students a double-track program in mathematics is offered in the ninth grade. This of course would not be typical of the very small high schools, which are often severely handicapped because of limitations of staff and room facilities. It may also be noted that, even for those schools reporting a double-track program, little uniformity was found among the courses other than algebra. About half of the ninth-grade students in these schools were taking algebra, about one-third were taking general mathematics of unspecified content, and about one-sixth were taking some more or less specialized courses such as commercial arithmetic, shop mathematics, or consumer mathematics.[2]

The advisability of offering specialized courses such as these latter ones to ninth graders may be seriously questioned, though they could be entirely suitable for the later years of the senior high school. The ninth-grade students are generally too immature in actual experiences and too far from the real applications of such courses to attain anything like the full potential benefit from them. For the ninth grade it would seem that general mathematics as defined by the Commission on Post-War Plans would be preferable.

Aside from the planning of the courses themselves there seem to be three main problems at present associated with the double-track program in mathematics. The first is the almost total lack of really suitable textbooks prepared specifically for the second-track course. This is a situation which, it may be hoped, will be improved presently. The second problem, and it is a real and serious one, lies in the widespread tendency to feel that general mathematics does not have the same status of respectability as does algebra. This tendency can be observed among teachers as well as among students, and, while it is perhaps understandable, it is unfortunate. When teachers themselves disparage general mathematics, it can hardly be expected that the

[1] Second Report of the Commission on Post-War Plans, *op. cit.*, p. 208.

[2] Raleigh Schorling, What's Going On in Your School? *The Mathematics Teacher*, **41** (1948), 147–153.

students will become enthusiastic about it. If this attitude is to be corrected, it must be through an understanding of the ultimate aims of the two courses. If teachers can make students realize that algebra and general mathematics are simply parallel courses each of which is very much worth while but which differ because they are aimed at very different objectives, then there is a reasonable chance that general mathematics will come to suffer less by comparison. If this could come about, it would, in turn, do much to ease the third important and related problem of guiding the students individually into courses appropriate to their abilities and to their probable future needs. The extent to which such a condition can be realized will depend largely upon the attitude which teachers take toward the course in general mathematics.

Mathematics in the Senior High School. Some of what has been said about ninth-grade mathematics can apply with equal force to the mathematics of the senior high school. We may postulate that all students need training and practice in quantitative thinking, but it does not follow that in the later years the needs of all are identical in this respect. There are many capable high-school students who like mathematics for itself or who are interested in it because it seems to fit into their plans for subsequent work. Improved teaching and guidance in the junior high school could do much to increase their number. These students should be encouraged to pursue the regular sequential courses as far as opportunity can be provided. But there are also a good many others who either actively dislike or are indifferent to the sequential courses or who lack the ability to pursue these with understanding. Even by the end of the ninth grade some of these students still will not have attained the "functional competence" to which reference has been made. Clearly the mathematical needs of such students lie in a different direction. Just as in the ninth grade, no single course can satisfactorily serve the needs of all the students.

It thus seems apparent that the double-track program proposed for grade 9 should be continued in the senior high school, to the extent that *every* student would have the opportunity to attain reasonable literacy in the mathematical ideas, relations, and processes commonly used in the everyday life of the average adult person. This does not imply that parallel series of courses should run concurrently through all the years, but that in the senior-high-school program there should be offered at least one second-track course in general mathematics in addition to the regular program of sequential courses. This is in line with the position taken by the Commission on Post-War Plans:

. . . *New and better courses should be provided for a large fraction of the [senior high] school's population whose mathematical needs are not well met in the traditional sequential courses.*[1]

The Commission goes even further and suggests that

The content of this mathematics would clearly embrace substantial materials from at least several of the following areas: mathematics as related to trades and shop work; commerce and business; industry; agriculture. It is also clear that every pupil is potentially both citizen and consumer; hence all pupils should be given some understanding of the persistent problems that confront most of our families; viz, social security, taxation, insurance against the numerous hazards of life, and ways and means of stretching the dollar in order to buy the maximum of material comforts and values with a given income.[2]

One such course, based upon satisfactory completion of a three-year junior-high-school program, recommends 20 units centered about applications of percentage; insurance and investments; mensuration; intuitive geometry; elementary algebraic processes including graphs, formulas, linear and quadratic equations; and numerical trigonometry.[3] Work of the kind being done at the various institutes and conferences throughout the country should be of great value, not only in providing suggestions of detailed content suitable for such courses, but also in providing vision, skill, and inspiration for those who will teach them.

It would be unfortunate, of course, for enthusiasts to make unwarranted claims about the advantages of such courses. For example, students should never be given the impression that they could take the place of the sequential courses in algebra and geometry as preparation for work in college mathematics. Such a course should not sail under false colors. Teachers and guidance counselors should make clear its limitations as well as its advantages. It should always be represented to students for just what it is: a course intended to be useful by providing better understanding of common quantitative matters that many people encounter in their everyday lives. In this way misunderstandings and disappointments can be avoided, and the course will command added respect.

The inclusion of a second-track course could, in addition to having its own direct values, react favorably on the sequential courses as well.

[1] Second Report, *op. cit.*, p. 210.

[2] *Ibid.*, pp. 210–211.

[3] J. W. McClimans, Functional Units of Instruction in Senior Mathematics, *Contribution to Education* 275, (Nashville, Tenn.: George Peabody College for Teachers, 1940).

Too often in the past, when these alone have been offered, they have been diluted or emasculated in an effort to fit them to the abilities and the interests of all the students, even the poorest and least interested ones. If these courses can be reserved for those students who will be able really to profit from them and who will have a real interest in them, then they will not need to be diluted but can be made to serve their real purpose, *viz.*, the development of genuine mathematical power and understanding.

Algebra, demonstrative geometry, and trigonometry will doubtless continue to make up the regular sequence of courses in the senior high school. Efforts to fuse them have not been notably successful, and it is almost certain that for the most part they will continue to be given as separate subjects. A great deal has been done in the past half century to improve these courses to make them more meaningful and appropriate for high-school students. Among the large and important contributions in this direction the following may be mentioned: the attention given to the formulation of real functional objectives; the re-examination of the content in the light of these objectives; reduction of the definitional approach and of sheer formalism; the elimination of much meaningless or unnecessary detail; the increased emphasis on the understanding of broad ideas and relations; rearrangement of internal topical sequences and reallocation of emphasis for better effect; improvement of problem material; the general resetting of the work to facilitate learning and understanding by relatively immature students.

But in spite of all the improvement that has been made in the sequential courses, much remains to be done.[1] If these courses are to fulfill their main purpose, there must be even more emphasis upon major concepts and principles in order that the structure of the courses may have both focus and continuity. Details should always be linked to these basic ideas, for only in that way can the large understandings be strengthened and clarified, and only through these relations can the details themselves take on full meaning. Continual efforts should be made to provide better illustrations and simple applications of these concepts and relationships. More and better use ought to be made of the various multisensory teaching aids which are becoming increasingly available. The new textbooks which appear will probably exhibit a continual trend toward such improvements as have been mentioned. But any textbook can be helpfully supplemented by suggestions gleaned from other textbooks or from other sources. We have no

[1] See W. D. Reeve, Significant Trends in Secondary Mathematics, *School Science and Mathematics*, **49** (1949), 229-236.

right to be complacent. The quest is endless. Every teacher of high-school mathematics ought to be always sensitive to the need for improving these courses and always alert to find ways in which they can be improved.

Mathematics in the Junior College. The junior college is at once a terminal school and a preparatory school: terminal for those students who for one reason or another do not expect to pursue advanced and specialized work in the college, university, or technical school; preparatory for those who do go on to take such work. It seems clear that the mathematical needs of the terminal-type students will generally be different from the needs of those who are to go into advanced work for which the usual sequential college mathematics is prerequisite. The customary mathematical program for these preparatory students is in general adequate and suitable to their needs. It consists, as a rule, of either separate or integrated courses embracing college algebra, plane trigonometry, analytic geometry, and a year's work in calculus. These courses, however, are not suited to the backgrounds, the interests, or the vocational needs of most of the terminal-type students in the junior colleges. Although these students have long made up a large and increasing part of the enrollment, it was not until around 1935 that much effort was made to provide special mathematics courses really planned for them.

On the other hand, after the movement got started it proceeded at an accelerated pace, and a sampling study indicated that by 1940 almost half of the colleges and junior colleges were offering some sort of course in general or "cultural" mathematics for nonspecializing students.[1] The recommendations in the report of the Joint Commission in 1940 probably gave added impetus to this movement. This report, in the interest of flexibility, presented two different topical outlines as being suggestive of types of organization which might be suitable. Each of these presupposed only one year of high-school algebra and one year of plane geometry, and each was pointed toward a broadened view and an enriched appreciation of mathematics. One of these outlines emphasized a utilitarian bias on the higher levels, as well as general educational value. It included such topics as measurement and computation, elementary trigonometry, graphs and equations, conic sections and some of their applications, statistical concepts, elementary mathematics of finance, series, the nature and simple applications of derivatives, and the concepts and uses of integration.

[1] Kenneth E. Brown, Is General Mathematics in the College on Its Way Out? *The Mathematics Teacher*, **41** (1948), 154–158.

The other outline was in the nature of an orientation course planned not primarily from the standpoint of external applications of mathematics but mainly with a view to giving students a more or less philosophical or theoretical overview of the nature of the whole field of mathematics. In the main it is concerned with basic mathematical concepts and theoretical considerations. It emphasizes the critical examination of the fundamental nature of certain primary concepts, of the nature and significance of implication, and of the internal structure and consistency of certain portions of mathematics.[1]

There is little uniformity in the content or arrangement of the numerous textbooks which have been published for general mathematics at the college level, and the problem of selecting a textbook is difficult. It places much responsibility on the teacher, since it requires careful comparison and appraisal of widely different books. Indeed, many teachers feel that no single one is satisfactory and prefer to use material selected from several textbooks.[2]

There can be little doubt that interest in the problem of general mathematics for the nonspecializing student is still high and perhaps still increasing,[3] but it cannot be said that the problem is rapidly approaching any well-crystallized solution yet. Indeed, it would appear that, in spite of widespread interest, general mathematics appears to have lost ground in the period from 1942 to 1947. Brown has reported that, out of more than 200 colleges and junior colleges which had offered such courses, only 60 per cent were continuing to offer them in 1947. In 40 per cent of these schools the courses had been found unsatisfactory and had been discontinued. The main reason given for this was that no satisfactory textbook could be found. In fact, only 30 per cent of the schools reported that the textbook being used was regarded as satisfactory. Criticisms were that the textbooks were too difficult, that the applications were beyond the experience of the students, that the books were sketchy and superficial, that they contained topics in which the students were not interested, and that they were not written in such a way as to stimulate the interest of the students.[4]

We should recall that the foregoing discussion applies to courses for which two years of high-school mathematics are prerequisite. Up to this time not much has been done to provide courses for those college

[1] Joint Commission, *op. cit.*, pp. 153–154.

[2] Kenneth E. Brown, What Is General Mathematics? *The Mathematics Teacher*, **39** (1946), 329–331.

[3] Commission on Post-War Plans, Second Report, *op. cit.*, pp. 213–215.

[4] Brown, Is General Mathematics in the College on Its Way Out? *op. cit.*

students who lack this prerequisite, and though some experimentation along this line is going on, the nature of whatever may develop from it must still lie in the field of conjecture.

If general mathematics courses for nonspecializing college students are to have a fair chance to establish their validity and worth, there needs to come about more widespread agreement on definitive statements of aims and of topical content for the courses. Such agreement would pave the way for more satisfactory textbooks and would cut down the present confusion about what should be included in the courses. Unless greater agreement does come about, the confusion is likely to persist, and as long as it does persist, the courses in general mathematics will never command the same respect as that which is accorded to the regular sequential courses.

The Need for Guidance. The character of the appreciations and the intellectual interests which people acquire is, to a large degree, the product of the environmental influences to which they are subjected during the years of school life. Certainly there is room for the light and the trivial, and life would be dull without them. But they should supplement, rather than supplant, those experiences leading to the enduring satisfactions which lie on the high planes of intellectual and emotional life.

Surely, in this day of expanded curricula and free electives, it is scarcely necessary to argue the need for wise counseling and guidance of secondary-school students in the planning of their school programs, and every elaboration of the curriculum makes this need more acute. Too often these young people have been allowed to follow the line of least resistance, selecting courses which interest them at the moment and avoiding those which do not, with little regard to any underlying plan or long-range consideration. But if these young people are to be brought to intellectual and emotional maturity, their experiences must be planned on a more substantial foundation than that afforded by their own temporary and immature interests. Their interests need not only to be guided but to be awakened and expanded.

> For it must be remembered, as has been said before, that the school has an obligation to create capacities of one kind or another, and should explain to pupils the advantages which may result from them, though it recognizes that in many cases the capacities will not all be employed.[1]

Wise guidance and cultivation of the interests and tastes of the student in his early years will do much to avoid later feelings of inadequacy and frustration and to ensure his eventual attainment of

[1] Joint Commission, *op. cit.*, p. 207.

the enduring intellectual and emotional satisfactions which are characteristic of the cultured individual. Mathematics can make large contributions to this end. It offers a field unsurpassed by any other subject for sheer intellectual play, and in the hands of an inspiring teacher it may be made as fascinating as any game. That a high order of satisfaction may be derived from the prosecution of this intellectual play is not so generally recognized as it should be. The reason for this is that only too rarely have real stimuli been afforded and properly presented to the children. The trouble has been in the dreary way in which mathematics has too often been taught—not in the subject itself. Those concerned with the educational guidance of students should recognize that, as a school subject, mathematics has large potential contributions to make in the field of appreciations and intellectual satisfactions, in addition to its disciplinary and practical values.

It is of great importance, too, that those responsible for guidance make sure that their students are informed about the vocations and the academic and professional fields in which mathematics is helpful either as a tool for direct use or as a prerequisite for further study. Most people, even guidance officials and teachers, do not realize the extent to which mathematics enters into many vocational fields, and certainly a great many high-school students are uninformed about this. They are uninformed also, and at times even misled, about the various fields of academic and professional study for which formal high-school mathematics is prerequisite. As a result of this lack of information many students who have by-passed mathematics in high school have later found themselves either barred from going on into vocational pursuits or fields of study which they wished to follow or seriously handicapped in these pursuits. The disappointment and the frustration of plans thus occasioned could have been avoided in many cases through wise guidance by an informed counselor.

It was for the purpose of making this sort of information readily available to all concerned that the Commission on Post-War Plans devoted its fourth and final report wholly to the matter of guidance with respect to mathematics.[1] This report, covering 25 printed pages, is addressed directly to high-school students. It is written simply and

[1] Guidance Report of the Commission on Post-War Plans, *op. cit.*, pp. 315–339. (Bound reprints of this Guidance Pamphlet may be had for 25 cents each, postpaid, from The National Council of Teachers of Mathematics, 1201 Sixteenth St., N.W., Washington 6, D.C. In quantities of 10 or more the pamphlet will cost only 10 cents per copy.)

clearly and is couched largely in informal conversational style. It is easy to read, and it makes interesting reading. Above all, it contains a great deal of detailed and valuable information. The outline of its section headings has already been presented on page 48. The Commission's plan of reprinting this report in the form of a low-cost "Guidance Pamphlet," with a view to its wide dissemination, has been carried out, and the tremendous potential value of this "Guidance Pamphlet" is being recognized. Within one year following its publication, more than 25,000 copies had been distributed for placement in the hands of teachers, students, guidance personnel, and administrative officers of schools. Certainly every mathematics teacher ought to be familiar with it and to use it for the benefit of his students.

Another guidance pamphlet, prepared by a committee of the Michigan Section of the Mathematical Association of America, was issued in 1948. It is entitled "A Mathematics Student– To Be or Not to Be," and its purpose is to help high-school students and guidance personnel become informed about mathematical prerequisites for various courses and curricula in Michigan colleges and professional schools. "We want to eliminate the possibility that students will be refused permission to take college courses because they did not know they had to have previously taken some high school mathematics." While this was just a mimeographed pamphlet, and while it referred specifically to colleges in only one state, the fact that it was sponsored by a state section of the Mathematical Association of America is significant. It points to the fact that the mathematics departments in colleges and universities are becoming acutely aware of the need for better guidance in the high schools.

The values of mathematics are not, as a rule, of such a nature as to be attained informally or incidentally. Their acquisition generally requires serious and sustained application to subject matter characterized by sequential continuity and cumulative organization. Almost without exception the people who learn mathematics learn it in school, and almost never is it learned under any other circumstances. However, the demands which mathematics makes upon the persistent and serious application of those who study it cause it to be avoided, under the free-elective system, by many students who could profit largely from it but who prefer, rather, to fill their programs with less arduous and less substantial courses. Unless such students are wisely advised and encouraged to continue their training in mathematics and in other substantial subjects as far as their capacities will permit them to profit from these studies, we need not be surprised to see the enrollments con-

tinue to decline. Education is too large and too serious a business to be dominated by a superficial philosophy of evanescent interests and transitory values. The minimum must not be allowed to become the norm. Those whose responsibility it is to counsel with secondary-school students have an obligation to help their students see beyond the immediate present; to point out to them potential values which at the moment are perhaps not obvious; and to give them a preview of those insights, appreciations, higher satisfactions, and higher instrumental values that lie beyond the threshold of mathematical study.

Exercises

1. Explain clearly why, in organizing either a whole program in mathematics or a course within the program, the first step should be to define the main objectives.

2. Contrast the two points of view set forth in this chapter regarding the mathematical needs of people. To which of these views do you incline? Give your reasons.

3. Point out two major respects in which the Second Report of the Commission on Post-War Plans goes distinctly beyond earlier reports in the matter of clarifying the aims for mathematical instruction.

4. Explain why those skills and understandings which most people regard as the practical values of mathematics are more prominent in the general mathematics courses of the junior high school than they are in the later sequential courses.

5. In what way and for what students should the sequential courses be thought of as practical courses? Why can they not be regarded as practical courses for all high-school students?

6. The Check List in the Guidance Report of the Commission on Post-War Plans contains 29 items of mathematical attainment. Which ones of these, in your judgment, represent practical values? Which ones are drawn from seventh- and eighth-grade mathematics?

7. Functional competence in mathematics is defined in terms of this Check List. Does this seem consistent with the position that the main objective in the seventh and eighth grades is the attainment of functional competence?

8. The Commission on Post-War Plans contends that seventh- and eighth-grade mathematics should be a single unified program and that essentially the same work should be given for all normal students. Give and discuss the arguments used to support this position.

9. Why should a double-track program in mathematics begin with grade 9?

10. Defend or criticize the assertion that the widespread acceptance of a double-track program is the most significant recent improvement in the mathematical program for the ninth grade.

11. What particular circumstances have operated to retard the development of satisfactory general, or second-track, mathematics above the eighth grade, and why have the general courses not been more favorably received? Discuss this fully. Do these conditions still persist?

12. How can students, teachers, and parents be made to realize that general mathematics is just as "respectable" as the sequential courses?

13. Describe the handicaps encountered by small high schools in trying to improve their programs in mathematics, and explain what might be done to meet and overcome these handicaps.

14. Show how and why the effect of having a double-track program in the ninth grade and senior high school would react favorably on the work in the sequential courses.

15. Give reasons why the current interests of students cannot be relied upon to serve as a satisfactory guide for the selection of courses in high school.

16. From the list of 19 current issues given on pages 83 and 84 select the 5 to which you would accord top priority. Defend your selection.

17. In a high school offering a double-track program in mathematics, what criteria should be used and what information and circumstances should be taken into account in counseling individual students as to which course each one should take?

18. Defend or criticize the assertion that to make and apply generalizations involves transfer of training in a true and vital sense and that without the ability to generalize and apply ideas there can be no really "functional" learning.

19. What, in your opinion, is the most significant contribution which the Commission on Post-War Plans has made? Justify your answer.

20. Discuss the nature of the work carried on at the various mathematics institutes and workshops, and tell what kinds of benefits could be expected to result from it.

21. Make a table of chapter headings for what you think would be a good second-track, or general, mathematics book for grade 9.

22. Should a course in consumer mathematics be offered in the eleventh or twelfth grade? If so, what should it include? Should it be required for graduation from high school? Why or why not?

23. Should the attainment of functional competence as defined by the Check List in the Guidance Report be a requirement for graduation from high school? Give your reasons for your answer.

24. Make up a test based on that Check List and designed to test for the attainment of functional competence in mathematics.

25. Describe the present status and trend of general mathematics in the junior college, and try to account for it.

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CHAPTER V

MATERIALS OF INSTRUCTION: AIDS TO TEACHING

TEXTBOOKS AND WORKBOOKS

As a teaching device in most subjects the textbook occupies a unique place and performs a unique function. It is an extremely important feature of the American educational plan because it largely determines the content and organization of the courses of study in many subjects. This is particularly true of the courses in mathematics. Indeed, it is scarcely an exaggeration to say that in most cases the mathematics textbook *is* the course of study. The evaluation and selection of textbooks is therefore an important matter. Moreover, it is becoming an increasingly difficult matter. Modern educational theory and practice are in a state of flux, and textbooks are continually changing because of their high sensitivity to the frequent and sometimes radical variations of theory and practice. Departures from tradition range all the way from conservative modification to radical reorganization. Many new texts appear annually to add to the already bewildering array, and the rate shows no sign of decreasing. Accordingly it may be expected that the problem of textbook selection will become progressively more complex.

In recent years a large volume of supplementary material has appeared in the form of workbooks. These have less of tradition behind them than have textbooks, and approval of them is not unanimous. Few people would regard them as being commensurate with the textbooks in point of importance. Doubtless they are not all of equal intrinsic merit, and it is very probable that the same workbook will be more useful to some teachers than to others. Generally speaking, however, workbooks which are properly designed and which are used in appropriate situations and in appropriate ways embody certain features which may make them valuable aids in instruction. Where workbooks are to be used, the problem of deciding upon the one best suited to the requirements of a particular situation becomes a matter of genuine importance, and the teacher has a real responsibility in recommending the one to be selected.

On the whole, experienced teachers are probably more competent to

make comparative evaluations of textbooks and workbooks within their respective fields of specialization than any other individuals connected with the schools. This is recognized by administrative officers in many school systems. In large cities the task has been assigned frequently to committees of teachers, the younger and less experienced teachers often working with, and under the guidance of, the older and more experienced members of the staff. In smaller places where it is impossible to have such committees, individual teachers are often entrusted with the task and are charged with the responsibility of making recommendations to the proper administrative and supervisory authorities.

In view of its importance and its difficulty, the whole matter of evaluating textbooks and workbooks has received less than its due share of attention in the training of teachers. It is a responsibility which many teachers will be called upon to assume officially, and it is a matter to which all teachers will have to give some attention as new books come into their hands. The lack of some body of principles or criteria for evaluation is likely to leave one unduly impressionable to the arguments of commercial book salesmen or to the appeal of attractive sales literature.

Difficulties in the Way of Wise Selection. The major difficulties in the selection of textbooks and workbooks can be traced to the need for a valid and accepted body of definite criteria which take into consideration the fundamental issues involved, which are searching but which can be stated and applied objectively without becoming superficial, and which are not so cumbersome as to make their use prohibitive from the standpoint of time and labor. This, of course, is a large order. It is even possible that, in view of the nature of the problem, the conditions here imposed may be to some extent inconsistent or contradictory. Nevertheless it is only through an attempt to approximate this ideal that an intelligent and practical attack upon the problem of textbook selection can be made, and it is not the part of wisdom to ignore the matter even if it should appear that a *completely* satisfactory solution is improbable.

Other obstacles are encountered in the paucity of teachers trained in scientific methods of analysis and comparison and in the limited amount of time which most teachers have at their disposal for such activities. It is believed, however, that it is possible for these matters to be corrected more easily than the one discussed in the preceding paragraph. Given a reasonable amount of intelligence, patience, and willingness, analysis is a technique which can be learned. Some train-

ing in such work should be a part of the professional equipment of all teachers. Wise evaluation, of course, implies more than a mere mechanical sort of analysis; it implies familiarity with the aims of mathematical instruction, with the principles of appropriate selection and organization of subject matter, and with effective instructional devices. But these also can be learned. Admittedly it is not always possible for teachers to be granted time from their instructional duties for textbook evaluation, though this practice is found to some extent, especially in the larger cities. The limited amount of time which most teachers can give to the evaluation of textbooks and instructional materials only emphasizes the need for usable criteria of the nature indicated above.

Differences in Published Criteria for Textbook Evaluation. There have been various attempts to set up such lists of principles or criteria for the evaluation of mathematics textbooks. In the main these have been useful in focusing attention on the need for objective means of evaluation and comparison and in providing patterns of analysis. The great variation among these patterns, however, makes it clear that different people, presumably all of high competence, may have very different ideas as to the relative importance of various elements, and that they may also have very different ideas as to what elements should be included in such analytical comparisons. One of the shortest published lists of criteria for the selection of mathematics textbooks contains only six items, all of which are broad general considerations lacking definition or adequate explanation and all of which would require a highly subjective approach. The entire list occupies about one-third of a printed page in an ordinary sized book. At the other extreme is found a list which contains four general categories with 14 major headings under which are listed 86 items making up the complete analysis.

It is obvious that the first of these two lists would not suggest the detail necessary to a careful comparative analysis of textbooks, though it would certainly be better than having no criteria at all. It is equally obvious that the task of comparing any considerable number of texts from each of 86 different standpoints would involve such an immense amount of time and work that it could be undertaken only as a cooperative effort by a large group of teachers or else by some individual who might be free to devote his entire time and attention to the task over a considerable period.

There is another interesting contrast between these two lists of criteria. The shorter list is made up of statements of characteristics

each of which is considered to be a *sine qua non* for acceptability. On the other hand, the items in the longer list are in the form of questions, and there is no indication that any of the items represent characteristics regarded as absolutely essential. In other words, the short list seems to have been designed mainly for the purpose of *eliminating* books which fail to measure up to all its criteria, while the longer list apparently is planned more definitely for purposes of *comparison.*

The Score-card Type of Comparison. In neither of these two lists of criteria is there any indication of relative values among the different items such as may be found in some other lists of the "score-card" type.[1] Such lists have the advantage of taking into consideration the relative values of the different characteristics of textbooks and of presenting summaries of the comparisons or analyses in a compact quantitative form which lends itself easily to objective comparison. On the other hand, there are limitations which should be taken into account in viewing the results of the use of such scoring devices. In the first place, the maximum number of points that may be assigned to each different item must be either arbitrarily fixed or else determined by the combined judgments of two or more people. Also it should be borne in mind that considerations to which a given degree of importance is attached in certain situations may be regarded as more important or less important under other circumstances. Finally it must not be forgotten that, although the score card may be arranged in such a manner that each item can be scored by assigning a number which indicates the contribution of that item to the sum total of the evaluation of the book, the number itself, while it appears to be perfectly objective, may have been arrived at through an entirely subjective judgment of the extent to which the characteristic in question is embodied in the book.

In spite of these limitations, however, the score-card method of evaluating textbooks is useful, and it has the merit of great flexibility and adaptability. A score card can be designed to include any points which may be regarded as important and can be made in just as great detail as may be desired. A number of such score cards have been published, but, in cases where none of these are available or seem to meet the needs of a particular situation, groups of teachers or even individual teachers can construct others to suit their own requirements.

[1] See Edwin S. Lide, Instruction in Mathematics, *Bulletin* 17, 1932, National Survey of Secondary Education, *Monograph* 23 (Washington: Government Printing Office, 1933), p. 58; also E. R. Breslich, "The Administration of Mathematics in Secondary Schools" (Chicago: University of Chicago Press, 1933), pp. 134–135.

The main considerations are that the items should be regarded as important in the given situation; that as far as possible they should be amenable to objective examination, evaluation, and comparison; and that they should be conducive to economy of time and labor insofar as may be compatible with the degree of completeness and penetration desired.

Things to Be Considered in Selecting Textbooks. The following list suggests a number of items which are often regarded as being important in judging the merits of mathematics textbooks. In presenting this list of items no attempt has been made to set forth an exhaustive catalogue of all points which might be considered. Not all the items listed can be treated with the same degree of objectivity. No attempt has been made to classify the items or to suggest relative degrees of importance, and the order in which the items are listed is of no significance. It is believed, however, that the list may be suggestive and helpful to those who may wish to prepare rating schedules.

1. Date of copyright
2. Experience and qualifications of the authors
3. Direct practical or social usefulness of the content
4. Cultural or disciplinary values of the content
5. Mathematical consistency in the organization of the subject matter
6. Appropriateness of the vocabulary to the age level of the students for whom the book is intended
7. Adequacy and proper gradation of the exercises
8. Omission of obsolete material
9. Aims of the authors as stated in the preface
10. Provision within the text for drill, review, and maintenance
11. Provision for individual differences in ability
12. Provision for individual differences in interests
13. Adequacy of development of concepts
14. Style of writing
15. Devices and material for motivation
16. Instructional aids
17. Suggestions and devices to improve study habits
18. Organization of subject matter into psychological units
19. Distribution of exercises
20. Appropriate distribution of emphasis on different topics
21. Adherence to, or departure from, the traditional order of major topics or units
22. Innovations in treatment of topics or units
23. Conformity with recommendations of important committees
24. Explanations and definitions
25. Character of the problem material
26. Illustrations (pictorial or graphic)
27. Provision for field projects and supplementary work

28. Typography and paper

29. General appearance of the book, especially the durability of the binding and attractiveness of the format and cover

30. Price

Each item which is to be used in a textbook score card should be accompanied by an explanation defining as clearly and as objectively as possible the precise meaning and scope of the item and defining various degrees of merit with reference to that item.[1]

Function and Use of Workbooks in Mathematics. The function of the workbook in mathematics is more specialized than that of the textbook. Workbooks are generally used as supplementary teaching devices and almost never as the basic body of material for a course. They are built with the idea of suggesting appropriate activities and exercises rather than with the idea of providing information, and most of them contain little or no developmental material. Workbooks are merely specialized and elaborated forms of a teaching device that has been in use as long as mathematics has been taught from books, *viz.*, the suggestion of problems and activities through which it is expected that students will acquire the desirable skills, understandings, abilities, and appreciations which are the objectives of the work.

As a rule it has been necessary for students to recopy problems and exercises from their textbooks before solving them. While they probably derive some benefit from this practice, such copying is in general not an educational activity. It is usually wasteful of time and energy, and it not infrequently results in mistakes. The practice of having students copy exercises from their textbooks also makes heavy demands upon the time and energy of the teachers in the checking of student papers, because it is almost impossible under the circumstances to get students to prepare their papers in a uniform manner which will be conducive to speed and ease in checking. The workbook eliminates the necessity for copying problems and exercises, and in this way it saves time for the students and prevents them from making mistakes

[1] For examples of such definitive explanations see the following references:

L. E. Mensenkamp, Some Desirable Characteristics in a Modern Plane Geometry Text, *Fifth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1930), pp. 203-206.

Manning M. Pattillo, The Selection of Books in the Field of Mathematics, *School Science and Mathematics*, **43** (1943), 468-475.

Cecil B. Read, Selection of Mathematics Texts, *ibid*, **42** (1942), 809-812.

D. E. Smith and W. D. Reeve, "The Teaching of Junior High School Mathematics" (Boston: Ginn & Company, 1927), pp. 205-215.

that may have no relation to their ability to perform the required mathematical tasks. It provides a uniform pattern for the written work of all the students, and in this way it conserves the time of the teacher, diminishes the labor involved in checking the papers, and increases the objectivity of the marking.

Some workbooks are so arranged as to provide programs of systematic cumulative review and maintenance work, and there is considerable evidence that really good books of this type can be used economically and very successfully for this purpose. Stone, for example, has reported experimental evidence which seems to indicate clearly that the use of one particular workbook definitely increased the effectiveness of instruction in junior-high-school algebra classes, concluding that

1. Independent of subject matter and group, the use of the workbook achieved better results.

2. Independent of subject matter and group, the use of the workbook held the class together better in line of thought.

3. Independent of subject matter and group, the workbook produced more high scores and fewer low scores than the textbook.[1]

He also states that the students preferred the workbook to the textbook and gives a list of 13 advantages claimed by teachers for the use of the workbook in mathematics. Other investigations also have yielded results similarly favorable to the use of workbooks.

Objections to Workbooks. The advantages which the workbook offers have caused it to have a rapid rise in popularity, and it appears to have become a permanent fixture in the American scheme of instruction. However, as an instructional medium, it has not been without its critics. There are those who contend that some workbooks in mathematics have not been prepared with foresight and care. The preparation of a *good* workbook cannot be done offhand. Years of painstaking work and research have gone into the making of some of the better workbooks, but others have appeared which give evidence of superficial planning and careless workmanship. The publication of such books tends to produce an unfavorable attitude in the minds of many people toward all workbooks.

Another objection has been that the exercises are so uniformly objective that they tend to make the students into mere blank fillers and give no training in the complete organization and expression of ideas. This objection is in part a valid one, but it overlooks two com-

[1] Charles A Stone, The Workbook in Mathematics, *School Science and Mathematics*, **35** (1935), pp. 382–387.

pensating factors. Workbooks in mathematics are likely to involve a smaller proportion of strictly informational exercises than workbooks in some other subjects, and they are, after all, but supplementary to the regular textbooks, and do not by any means represent the totality of the materials of the course.

A third objection to the use of workbooks is that they involve extra expense. This cannot be denied. Not only do they represent an extra initial outlay, but they cannot be resold and used again as is often done in the case of textbooks. No workbook is of any use except a new one. In this connection, however, it should be considered that the cost of workbooks is generally small in comparison with the cost of textbooks and that the additional expense involved in the purchase of supplementary workbooks may easily be justified if it results in more effective learning.

Some Characteristics of a Good Mathematics Workbook. In order to be thoroughly satisfactory, a mathematics workbook should embody the following characteristics:

1. It should aim to be instrumental in bringing about certain definite educational outcomes. These outcomes should be specified by the authors of the workbook in its preface, and the materials of the book should be prepared and arranged in such a way as to contribute definitely to the attainment of these outcomes.
2. Space should be provided for most of the written work to be done in the book itself.
3. The content should supplement that of the basic textbook and should coordinate well therewith.
4. It should economize the students' time by minimizing or eliminating the need for copying exercises and problems.
5. It should economize the time and minimize the work of the teacher in checking the written work.
6. Its exercises should make more than superficial or trivial demands upon the students.
7. It should provide opportunity and incentive for each individual student to work at his own optimum rate.
8. It should provide some simple arrangement whereby each student can keep a record of his own achievement and progress.
9. It should be accompanied by a list of answers to all exercises to facilitate checking. If these are not in the book itself, there should at least be available a complete key for the teacher.
10. It should have a pleasing format and clear typography. In particular, the type face should be large enough to be easily read.
11. It should have a substantial and appropriate cover.

12. It should be bound in such a way that, when opened, it will lie flat on the desk.

13. It should be moderately priced.

Workbooks Combined with Textbooks. In a few cases authors have prepared books in mathematics which combine in a single volume the features of the basic textbook and those of the workbook. These are not just textbooks with the workbooks attached as appendixes. The two are woven together and coordinated at all points. The main difference between these books and the usual textbooks is that in these combination books the exercises are generally put up in workbook form with space provided so that most of the written work can be done in the book rather than on separate paper. These books are necessarily longer and wider than the usual textbooks, just as is the case with practically all workbooks. They are also bound as workbooks are, *i.e.*, less expensively and less permanently than the usual textbooks. They cost only about half as much as regularly bound textbooks in the same subjects, but they have no resale value since they cannot be used more than once.

Theoretically this sort of combination of textbook and workbook should combine and enhance the advantages claimed for both of these separately, because it may be reasonably presumed that the authors are in a better position to make an effective correlation of the developmental work of the textbook and the exercise material of the workbook than the average teacher. The limited information which is available indicates that students like the combination type of book. There is every reason to believe that books so organized are pedagogically sound, and the fact that it has been found possible to produce them at a low price is also in their favor. The principal objections to them seem to be that the low price has necessitated binding them with destructible covers and that, after having been used once, they cannot be used again. These features probably account for the fact that such books have not come into more general use in the schools.

EQUIPMENT FOR MATHEMATICAL INSTRUCTION

Equipment for mathematical instruction falls into two classes: that which the student needs in order to pursue his own individual study, and that which can be used in common and which need not be provided individually for each student. The first of these categories includes such obvious necessities as textbooks, writing equipment, simple drawing and measuring instruments, and in some cases special instruments

such as the slide rule. These items may be designated as the personal equipment of the student. They constitute minimum working equipment without which any effective work in mathematics is impossible.

To many people it may appear that any discussion of mathematical equipment beyond this minimum is beside the point. It is an unflattering commentary that school purchasing officers and administrative officers not infrequently take the attitude that since mathematics is not listed among the laboratory subjects, it requires no equipment other than the items already mentioned. This point of view, however, is not consonant with modern educational thought nor is it abreast of current practice in the more forward-looking schools. It overlooks the enrichment of the courses which may be made possible through the use of suitable equipment to relate the mathematics of the classroom to the mathematics of science, of commerce and industry, and of everyday life. It is desirable that such possibilities for the enrichment of mathematical work become more generally realized and be put into practice.

The Equipment of the Individual Student. The equipment needed by individual students in mathematics is not extensive; rather, it is simple and inexpensive. It is very important that each student should have his own individual equipment. Most work in mathematics will be written work which will be based upon material in the textbook. It is therefore of first importance that every student have his own textbook and that he keep himself supplied at all times with an adequate supply of paper and well-sharpened pencils. Most teachers prefer to have each student provide himself with a loose-leaf notebook in which he can keep a supply of different kinds of paper which may be needed. This will usually consist of unruled white paper and a uniform type of graph paper. Such a notebook is a very satisfactory device; not only does it keep the paper together in a neat and convenient fashion and provide a means for filing and retaining important written work, but the other mathematical instruments such as the compasses, protractor, and ruler which should form a part of the equipment of each individual student can be attached to the rings of the notebook and can thus be kept always instantly available.

The responsibility for keeping his notebook and its contents in order and readily available should be placed squarely upon the student. Borrowing paper, pencils, or instruments is a lazy habit which fastens itself rapidly upon children and which becomes progressively demoralizing. Yet to expect that children will carry their books and supplies to and from the classroom without ever forgetting them is to expect the

impossible. Occasional borrowing is inevitable, but borrowing should be discouraged in every way possible. In cases where it cannot be avoided, students should borrow from the teacher rather than from each other. The teacher therefore should be provided with a supply of paper, pencils, compasses, rulers, protractors, and any other items of individual equipment which students will be expected to use.

The student's mathematical instruments should be simple, substantial, and inexpensive. A small *ruler* containing both inch and metric scales is desirable. The ruler should be thin so that it can be perforated and attached to the notebook rings when it is not in use. *Compasses* consisting of one leg and a sleeve into which a pencil can be fitted to form the other leg are generally both satisfactory and inexpensive. It is desirable that the compasses be of such a design that they can be attached to the metal rings of the notebooks. *Protractors* are made in a wide variety of designs and sizes. The main considerations are that the protractors shall not be too small and that the division markings shall be plainly legible. Metal or celluloid protractors are preferable to cardboard instruments because of their greater durability.

Mathematical Equipment Needed by the School. In addition to the equipment for the individual students there are numerous special items of equipment which can be provided advantageously by the school for general class use. The feeling is rapidly gaining currency that elementary mathematics could profit greatly from the introduction of a substantial amount of field and laboratory work, and there seems to be no doubt that more emphasis on this type of work would do a great deal to popularize the courses and to make them more attractive to the majority of students. The mathematics laboratory is no longer a misnomer or a mere phrase; it has become in many places an established medium through which the courses have been given new meaning and interest.

Extensive laboratory and field work in mathematics cannot be carried on without instruments which are of such a nature that they cannot well be supplied by the individual students. Such equipment should be provided by the school insofar as circumstances will permit. Appropriations for such equipment should form as legitimate a part of the school's budget as appropriations for equipment and supplies for athletics, home economics, the fine and industrial arts, or the natural sciences. Mathematics teachers and supervisors have been very remiss about keeping this point of view before their administrative officers, and this fact probably accounts in large measure for the failure

of these officers to recommend expenditures for mathematical instruments and equipment.

Instruments for Field Work. There are a number of instruments which can be used to add interest to the work in mathematics by demonstrating a large variety of fairly simple applications of mathematics to practical field problems and by making these applications clear and meaningful to the students. It would be desirable to have all of them available, but much interesting work can be done even with a small array of instruments. Where it is not possible to secure all that are wanted at the outset, a small beginning can be made and additional items added to the collection from year to year or as circumstances permit.

The *angle mirror* is a useful and relatively inexpensive little instrument with which a great many interesting things can be done. It is used primarily as a quick, easy, and practical means of laying out right angles in the field and of locating the vertices of right angles whose sides pass through two fixed points. This latter property makes it possible to use the angle mirror to lay out in the field circles whose diameters are known. Junior-high-school children can learn easily and quickly how to use this instrument, and they take great delight in using it to find heights of buildings, to lay out baseball diamonds and tennis courts, to do elementary mapping of small areas by means of offsets, and to carry on other fairly simple field projects. For most effective use of the angle mirror a *Jacobs staff* should be provided for mounting the mirror. *Tapes* for measuring distances will, of course, be needed in connection with much of the work done with the angle mirror, and it will be found very helpful to have some *ranging poles* and steel *arrows* or marking pins. Homemade substitutes for the ranging poles and arrows can be used if necessary, but generally they are less satisfactory than the manufactured articles. Two or three angle mirrors together with the supplementary equipment which has been suggested will form a useful and inexpensive nucleus for the collection of instruments, and the amount of interest which can be injected into the work through the use of these instruments alone will repay the investment many times over.

The *plane table* and the *alidade* are used for elementary mapping and surveying. These instruments are extremely simple and easy to use, but surprisingly accurate work can be done with them; in fact they are used extensively for much of the small-scale mapping done by the U.S. Coast and Geodetic Survey. Through the use of these instruments small areas can be mapped either by the method of radiation from a

point or by the method of intersections of lines of sight taken from the two ends of an established base line. Such work opens up to the students new vistas of the applications of mathematics to practical problems and never fails to stimulate a high degree of interest.

Proportional dividers can be used, among other things, to enlarge or reduce maps, diagrams, or pictures. This simple instrument is based upon the principle of proportionality in similar triangles. While its empirical use is very simple, it is extremely fascinating even to students who have not yet entered the junior high school.

The combined *hypsometer* and *clinometer* is a simple device for finding angles of elevation and depression and for measuring distances and heights of objects indirectly by means of sines, cosines, and tangents of these angles. The approximate values of these functions can be read directly from the hypsometer to a degree of accuracy sufficient for most purposes. This makes the use of tables of functions unnecessary in many problems. This instrument can even be used to give an approximate determination of latitude. When combined with the plane table it can be used to measure horizontal angles, and thus it can be made to do fundamentally the two things which the transit does, though, of course, not with the same degree of precision. It can be used for many purposes in practical field problems in simple engineering or surveying and in numerous activities connected with scout work. It is extremely valuable in mathematics classes because of the clear, simple, and striking manner in which it illustrates the principles of indirect measurement.

The *level* is an instrument used in finding differences in elevation. It is necessary in contour mapping, general surveying, and much of the work of civil engineering, but its sole function is the establishment of planes in which the lines of sight will be horizontal. The *transit*, on the other hand, is an angle measurer, as well as a leveling instrument. It can be used to measure directly angles in both horizontal and vertical planes, and it thus provides a combined means for securing all data needed in any field problems in surveying or mapping or in the indirect measurement of distances. For all leveling work and for some other work with the transit, the use of a *leveling rod* is required. The use of leveling instruments and the transit enlarges the scope and increases the precision of the work which can be done by a class.

The *sextant* is a sort of refined and variable angle mirror. It can do all that an angle mirror can do and more besides. Its primary use is in navigation, but there are many field projects in mathematics in which it can be used to advantage, so that, while it is not an absolutely essential item of equipment, it can do much to enrich mathematical field work.

Instruments for the Classroom. The *slide rule* is one of the most interesting and important of all mathematical instruments. It provides a rapid means of multiplying and dividing numbers, of taking certain powers and roots, and of solving proportions. It can be used in certain trigonometric work also, and it is extremely useful in solving problems in chemistry and physics. Moreover, it is coming into widespread use as a computing instrument in commercial and industrial work. Its main advantage lies in the convenience and the extreme rapidity with which it can be used. Its principal limitation lies in the fact that the ordinary 10-inch rule gives results whose accuracy is limited to about three significant figures. This is sufficient, however, for many practical purposes. Experience seems to indicate that it is better not to introduce the slide rule until the latter half of the ninth grade. Its use requires a good knowledge of decimals, but beyond this no special mathematical knowledge is needed. The slide rule can be learned easily, and its practical usefulness, its simplicity, and its efficiency combine to render it a valuable and fascinating instrument to students. Not infrequently students are asked to provide themselves with individual slide rules as a part of their personal equipment. These are now available at a nominal cost. Large demonstration slide rules can be purchased reasonably, and one of these should form a part of the permanent equipment of every mathematics classroom for students of the ninth or higher grades.

The almost universal employment of *calculating machines* in commercial and industrial work makes it desirable for students to know something about their nature and operation, and students are invariably interested in seeing these machines work and in learning to use them. At least one calculating machine for demonstration purposes may therefore be regarded as desirable equipment for the mathematics department. Courses in mathematical or applied statistics are sometimes offered in the junior college, and in such courses the calculating machine becomes really an essential part of the departmental equipment.

For work in solid geometry a *spherical blackboard* is a valuable asset. Students often experience much difficulty in visualizing the relations of lines, points, planes, and portions of spherical surfaces and volumes to each other and to the sphere itself. The use of a spherical blackboard is a great help to them in this respect.

Models of geometric solids are valuable, but mainly for display and demonstration purposes and to help students learn to visualize the figures. Occasionally models are directly useful in teaching. The use

of such models in the demonstration of theorems should not be overdone, but the display and examination of models in the general discussion of the solids is generally helpful to students in clarifying their ideas about three-dimensional figures.

Blackboard protractors, compasses, and *rulers* are almost indispensable and should form a part of the equipment of every mathematics classroom. *Blackboard stencils* for certain of the most commonly used figures are useful in economizing time and in providing accurate blackboard diagrams.

Other classroom equipment, which is not often found but which would be distinctly worth while, includes such items as *drawing boards, T squares, draftsman's triangles, parallel rulers,* and the *pantograph.* The display case may well contain such items as models of geometric solids, special mathematical instruments such as linkages for various curves, graphs, diagrams of mathematical figures, models or pictures of ancient mathematical instruments, scientific equipment involving mathematical principles such as balances, weights, combinations of pulleys, gears, inclined planes, arrangements of levers, and a great variety of other things. The nature and size of the mathematical collection is limited only by the special interests of the students and teacher and by the facilities available for housing the collection.

Useful equipment can often be borrowed from other departments in the school. Such things as combinations of pulleys, mechanical models involving various lever arrangements, and measuring instruments of various kinds such as graduated volumetric measures and vernier and micrometer calipers for linear measurement are always to be found among the equipment of the natural science laboratories, while calculating machines can sometimes be borrowed for demonstration purposes from the commercial department. Such equipment as can be made available by suitable arrangement with other departments in the school should not be included in requisitions for mathematical equipment unless it is needed regularly in the mathematics classroom.

Homemade Equipment. Some of the equipment needed for use in the mathematics classes can be made by the students. Homemade instruments will not, in general, give such precise measurements as can be obtained by using commercially manufactured instruments, but for many purposes they are sufficiently precise. In most cases where extreme precision is not particularly important, homemade instruments have two notable advantages over those that are produced commercially. In the first place, students always take a certain pride in equipment which they, themselves, have made, and this adds to

their interest in using such equipment. But more important than this is the fact that children are more likely to understand clearly the fundamental principles upon which mathematical instruments are based if they make the instruments themselves. Most mathematical instruments are fundamentally very simple in principle. For example, the transit appears to many people to be a very complicated piece of apparatus. Fundamentally, however, it is merely a device containing two movable protractors in planes which are perpendicular to each other. Admittedly a surveyor's transit will give more precise measurements than a homemade instrument consisting of two protractors mounted to operate in perpendicular planes, but it may well be doubted that a commercial transit will make the basic principles of such measurement any clearer than the homemade apparatus.

Among the instruments and items of equipment which can be made by students and teachers may be mentioned the following:

INSTRUMENTS OF SUFFICIENT PRECISION FOR MANY PURPOSES

Blackboard protractors
Plane tables and alidades
Levels
Field protractors for plane-table use
Clinometers
Instruments for indirect measurement of distance by using the principle of proportion in similar triangles
Proportional dividers
Angle mirrors
Ranging pole
Arrows
Leveling rods
Plumb lines

MATHEMATICAL EQUIPMENT OTHER THAN INSTRUMENTS

Mathematical models
Bulletin boards and display cases
Cross-section blackboards
Stencils for geometric figures

It should not be forgotten, of course, that the construction of mathematical equipment can legitimately occupy a small portion, at best, of the students' class time. There are certain things which students can make but which require an amount of time that is disproportionate to the amount of value derived from the project. For example, the construction of a sextant or a slide rule of any high degree of precision would require several hours. This probably would not be justified,

unless the work were done outside of class hours and done from sheer interest on the part of the student, because little can be learned about the principles of these particular instruments that could not be learned equally well from a commercially made instrument.

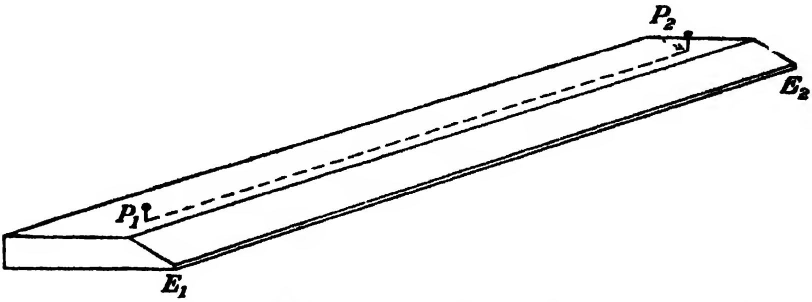


FIG. 2. Alidade This may be made from seasoned hardwood or clear white pine, cut and beveled as shown A good foot-rule with metal edge may be used

Pins P_1 and P_2 must be firmly set to be used as sights The line of sight P_1P_2 must be carefully tested to be sure it is almost exactly parallel to edge E_1E_2 This can be done by sighting a point 50 or 60 feet away If P_1P_2 and E_1E_2 are parallel, they will *both* sight the point without moving the alidade The pins must be set so that this condition will exist.

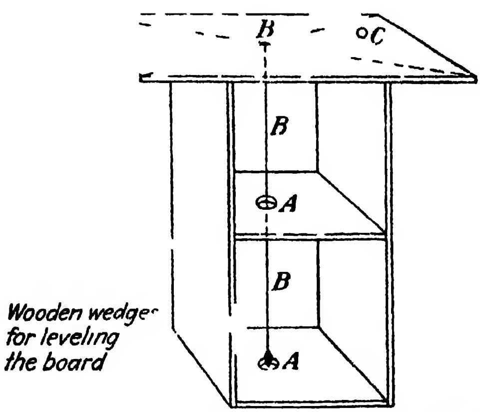


FIG. 3. Plane table, made from orange crate or wooden box with drawing board attached to one end by short screws from inside the box

A Two-inch holes measured to be directly under marked point *B* on drawing board *B*. Plumb line and bob suspended under point *B* on drawing board *C* Marble or ball bearing for testing level position of board

The function of the mathematics laboratory is not the same as the function of the industrial arts laboratory. The sole function of the mathematics laboratory is to provide stimulating and worth-while experiences touching mathematics and its applications. The practice of having students spend time in constructing mathematical equipment is justifiable only if it clarifies the understanding of the underlying

principles and increases the students' interest in the theory and application of the mathematics involved, or if it makes available equipment which would otherwise not be available and the value of whose contribution is at least commensurate with the time and work required for its construction.

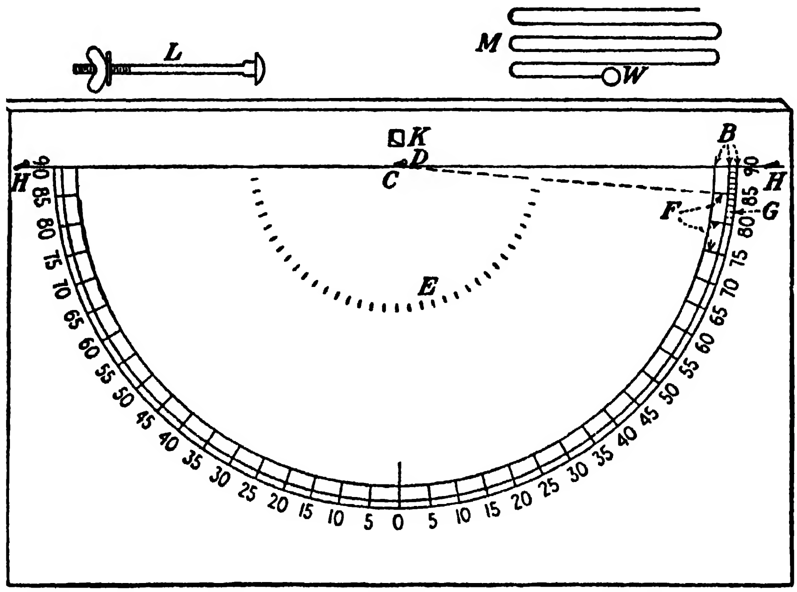


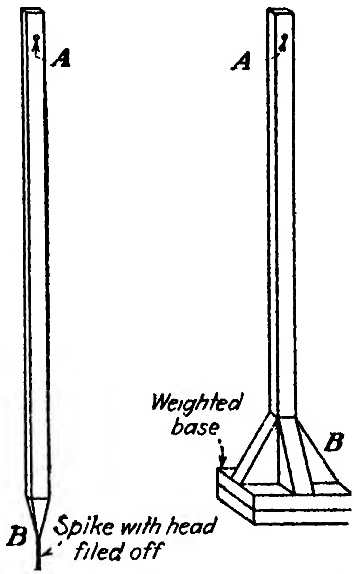
FIG. 4. Clinometer. *A*. Smooth piece of board, any convenient size (a piece 10 by 12 inches and ½ inch thick is very satisfactory). *B*. Semicircles described with compasses, using *C* as center. *C*. Center of semicircles; take distances *CB* any convenient length. *D*. Nail at *C* perpendicular to surface, for suspending plumb line. *E*. Pencil marks made at intervals of 5 degrees, using any fair-sized protractor centered at *C*. *F*. Extensions of radii through *E* marks. *G*. Five-degree divisions carefully subdivided into 1-degree divisions by estimate; continued around the semicircle. *H*. Pins perpendicular to surface on extensions of diameter; for sighting purposes. *K*. Hole for bolt to mount instrument on staff. *L*. Three-sixteenths inch bolt with washer and thumbnut for mounting instrument on staff. *M*. Plumb line; can be made from ordinary string with any convenient object, such as stone, for the plumb bob. *W*. Weight or plumb bob. The clinometer can be drawn: (1) Directly on the board. (2) On paper and mounted on board with paste or thumbtacks. (3) On tracing paper, from which multiple copies can be blueprinted for mounting on boards.

The Mathematics Classroom. The arrangement and equipment of the mathematics classroom itself can contribute much toward making possible effective instruction and, indeed, toward making possible a functional type of mathematics. Strictly speaking, mathematics is not an experimental science, yet it has an almost unlimited number of applications to economic, commercial, engineering, and industrial situations, and to the natural sciences. These applications are probably more interesting to most students than the abstract material of

the textbooks. If applications are selected and studied in such a way as to emphasize the mathematical principles which underlie them, they are valuable from a mathematical standpoint. They can never take the place of certain indispensable formal and sequential instruction in mathematics as such, but they can contribute a great deal toward clarifying the meanings of mathematical principles and toward supplying a powerful motive for the acquisition of understandings and skills. So far as possible the classroom should be equipped in such a way as to facilitate instruction along these lines. The following suggestions are offered in the belief that they may be helpful in this direction.

1. Desks for students are generally preferable to chairs with writing arms because there will be much "open-book" work, not a little of which will involve the use of drawing instruments. The desk top should be large enough to accommodate comfortably the student's open textbook and his writing and drawing equipment.

2. The room should contain adequate blackboard space including at least one coordinate blackboard. There should be blackboard equipment consisting of white and colored chalk, blackboard protractors, compasses and rulers, string, pointers, etc There should be a spherical blackboard for work in solid geometry.



A - Bolt hole for mounting clinometer
B - Alternate forms for base of staff

FIG. 5. Staff for mounting clinometer Any convenient height may be used

3. The room should be equipped with a demonstration desk or table for setting up experimental apparatus and with a work table where small groups of students may confer or work together when necessary. Bulletin boards, bookshelves, magazine racks, chart racks, storage space for instruments and equipment, files for papers and tests, and display space for models and instruments are all desirable features of the mathematics classroom. A pencil sharpener is indispensable.

4. The room should contain a mathematical library in which should be found a variety of collateral and supplementary textbooks in mathematics and related subjects as well as books and magazines for recreational reading to provide sheer enjoyment and to enhance the apprecia-

tion of mathematics.[1] Appropriate pictures relating to the history, progress, and application of mathematics will add to the general attractiveness of the room and will tend to induce a feeling and atmosphere favorable to the study of this subject.[2]

Exercises

1. Discuss fully the educational advantages to the teacher (especially the beginning teacher) of examining and comparing a considerable number of textbooks in different branches of secondary-school mathematics.

2. Take a recent textbook in seventh-grade mathematics or ninth-grade algebra, and read very carefully the author's preface. Then, after a careful examination of the book, decide whether the author has achieved the things which he set forth in the preface.

3. Examine several textbooks in plane geometry, and compare them with regard to format, typography, paper, binding, and general appearance. Which do you like best? Why? To what extent is this a matter of real concern in selecting a textbook for use?

4. In this chapter there are listed 30 considerations that are of more or less importance in the evaluation of textbooks in mathematics. Select the 10 which you think would lend themselves best to objective rating.

5. Select the 10 which you think are least objective.

6. Select the 10 which you think are most important, and justify your selection.

7. Compare two textbooks in seventh-grade mathematics, and decide which you think is written in the more interesting and readable style for students in that grade.

8. Examine carefully two textbooks in ninth-grade algebra, and decide which one you think is best with respect to the explanation of new topics.

9. Compare two textbooks in plane geometry, and decide which makes the more adequate provision for teaching devices such as illustrative examples, suggestions for study, reviews, self-tests, and the like.

10. Assume that you have prepared a score card for rating ninth-grade algebra books. Do you think this same score card could be used satisfactorily for rating plane geometry texts, or would it need to be modified for this purpose? Give reasons for your answer.

11. Enumerate the main advantages claimed for workbooks in mathematics.

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[1] The next chapter contains a list of references appropriate for the mathematics classroom library.

[2] For an amplified discussion of the equipment and arrangement of mathematics classrooms the reader should consult the following references:

Fred L. Bedford, Planning the Mathematics Classroom, *The School Executive*, **55** (1936), 290–292; Designing the Mathematics Classroom, *American School and University*, (1946), 199-202.

F. H. Gorman, What Laboratory Equipment for Elementary and High School Mathematics? *School Science and Mathematics*, **43** (1943), 335–344.

Raleigh Schorling, "The Teaching of Mathematics" (Ann Arbor, Mich.: Ann Arbor Press, 1936), pp. 81–86.

Jessie Roselle Smith, A Mathematics Workroom for the Senior High School, *The Mathematics Teacher*, **38** (1945), 126–129.

12. Examine several workbooks in mathematics for some particular grade or subject, and select the one you like best. What features of this workbook seem to you to be especially good?

13. Take some recent textbook for one of the courses in junior- or senior-high-school mathematics, go through it carefully, and write out a list of specific criticisms of the book.

14. Give a review of Gorman's article on laboratory equipment for mathematics classes. (The article is listed in the bibliography at the end of this chapter.)

15. Make a list of instruments which you believe should be a part of the permanent equipment of a well-ordered mathematics classroom or department in a junior-senior high school of about 600 students.

16. What items of equipment other than the instruments listed in the preceding exercise ought, in your opinion, to be provided in a modern mathematics classroom?

17. Name some items of equipment for mathematics classes which could be made by students at home or in the school shop, and describe how each of these could be made.

18. Name some items of equipment which would be helpful at times in mathematics classes but which could probably be borrowed from other departments in the school at such times as they might be needed.

19. Explain and illustrate the advantages that may be derived from such equipment as bulletin boards, display and exhibit cases, collections of models, and classroom mathematics libraries for students.

20. Select 10 good books and 2 good periodicals which you would recommend as a start for a classroom or departmental library in mathematics. In this connection you may wish to consult the following chapter.

21. What arguments would you present to your high-school principal in support of your request for a $200 appropriation to procure equipment for your mathematics department?

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CHAPTER VI

STIMULATING AND MAINTAINING INTEREST IN MATHEMATICS

It may be taken as axiomatic that students will work most diligently and most effectively at tasks in which they are genuinely interested. To create and maintain interest becomes, therefore, one of the most important tasks of the teacher of secondary-school mathematics. It is also one of the most difficult problems which the teacher encounters. The interest of students of secondary-school age is capricious. It is easily caught by any new thing, but it is as easily distracted to other new things. Ordinarily it cannot be depended upon to maintain itself for any great length of time unless the work has been carefully and deliberately planned to this end, and even then it usually needs occasional stimulation. Thus the motivation of the work in mathematics has two aspects, *viz.*, that of creating or arousing interest and that of maintaining the interest after the novelty of the work in hand has worn off.

As a rule, students readily become *interested* in things which are new or exciting, in things for which they can perceive practical values or applications to situations and fields of study in which they are already interested, and in things which involve puzzle elements or elements of mystery. Other things being equal, the possession of a background of related information tends to intensify interest in new work, but this is neither a necessary condition nor a sufficient guarantee for the awakening of interest. Novelty is sometimes more compelling than familiarity. The elements of novelty, of usefulness, and of sheer intellectual curiosity are the primary stimuli for the awakening of interest.

It is easier to interest students in their work than it is to keep them interested after the work has got under way and the novelty has worn off. In this connection it is worth while to observe that students tend to *remain interested* in those things which they can do most successfully and which they understand most completely. Inability to understand or to perform satisfactorily usually begets a condition of listlessness, inattention, and general loss of interest which not infrequently ripens into open disaffection. This is not to say that the work should be

made flabby and diluted and that it should never present difficulties to the students. Nothing can cause interest to flag more quickly than this, and nothing could be more undesirable from an educational standpoint. The work should present a continual challenge, but it should be a challenge in a real sense and not merely drudgery at tasks devoid of meaning or inexcusably difficult. Consequently, it is of the greatest importance that work in mathematics be so organized and conducted as to emphasize the values and the inherent intellectual challenge of the subject and to ensure understanding and a reasonable degree of competence by keeping the subject matter and the activities at a level of difficulty appropriate to the intellectual maturity of the students. Within these conditions are to be found the motives basic to hard and effective work in mathematics. Interest in the subject can be effectively augmented by numerous special devices and activities, some of which can be used in connection with class instruction while others, such as mathematics clubs and special programs, are essentially coordinate and supplementary activities.[1]

Motivation through Intellectual Curiosity. Many teachers and textbook writers have never recognized the power of sheer intellectual curiosity as a motive for the highest type of work in mathematics, and as a consequence they have failed to organize and present the work in a manner designed to stimulate the student's interest through a challenge to his curiosity. A notable instance of this is to be found in the fact that practically all theorems of demonstrative geometry are set up as exercises in establishing certain prestated conclusions rather than as exercises for free exploration and investigation of the consequences of certain hypotheses. Thus the element of discovery of the central fact or relationship in each theorem is removed at the outset, whereas this element of discovery could in many cases be retained and used to quicken the interest of the students. For example, the statement "Prove that an inscribed angle is measured by half of its intercepted arc" sets forth a task to be performed while the question "What relation, if any, exists between the number of degrees in an inscribed angle and the number of degrees of arc in its intercepted arc?" is designed to whet the curiosity of the student instead of satisfying it at the outset.

As a rule, secondary-school students are not intellectually lazy. That they may often appear to be so is due in large measure to the fact that their work is so largely set up for them as tasks to be done rather than as situations to be investigated. The average student in the

[1] Margaret Joseph, The Factor of Interest in the Teaching of Mathematics, *School Science and Mathematics*, 40 (1940), 201–207.

secondary school is an incurably curious individual, and the range of his potential intellectual interests is practically unlimited. The writers of modern detective fiction have recognized this and have capitalized on it. The enormous popular response which has brought this type of fiction to the front rank in sales and library withdrawals can be attributed only to the general reader's insatiate curiosity in the development and denouement of a problem situation.

Mathematical situations lack, of course, the lurid "human interest" of the ordinary mystery novel, but they do not lack the essential curiosity-provoking possibilities. "Think-of-a-number" games are popular at parties, even among people who anticipate or recall the study of algebra with dread, yet the games are nothing except algebra somewhat obscured, perhaps, by a screen of mysticism which only serves to stimulate curiosity. People are interested in seeing how numbers behave, and algebra is essentially the science of the behavior of numbers. Puzzle problems in mathematics have often been criticized as being "unreal" or "having no genuine application to life situations." A little experience in teaching algebra, however, will soon convince the most skeptical critic that problems do not need to represent "real" situations in order to be interesting to students. As a matter of fact, it is quite possible that the presence of the puzzle element in problems is often a greater stimulus to *interest* than those elements of so-called "reality" which are usually incorporated in the problems.

Obviously there must be system and organization in mathematics. Arithmetic and algebra cannot and should not consist entirely of number games and puzzles nor demonstrative geometry of incidental and undirected investigations. These are sequential subjects and must be developed in sequential form. Haphazard or piecemeal work will achieve nothing of value. But within the framework of the systematic organization of a course in mathematics at any level of secondary instruction there are many opportunities for motivating the work by deliberate stimulation of the curiosity of the students along the lines indicated. The greater the extent to which this is done, the greater will be the interest, understanding, and assiduity with which the students will work and the more meaningful and worth while will the work become to them.

Motivation through Application to Other Fields of Study. The relation of mathematics to other fields of study often provides an important means of stimulating interest. At all levels of secondary education from the junior high school to the junior college the contribution which

mathematics has made and can make to the more adequate development and understanding of many subjects is coming to be recognized more fully than ever before, and teachers should not fail to stress its importance from this standpoint. There is no lack of material bearing upon this matter. Many articles discussing the relation of mathematics to other fields of study can be found in such magazines as *The Mathematics Teacher* and *School Science and Mathematics* and in some of the yearbooks of the National Council of Teachers of Mathematics as well as in other miscellaneous publications.

The basic importance of mathematics in relation to other fields of study has been nowhere more emphatically or more strikingly portrayed than in the mural entitled *The Tree of Knowledge* which was displayed in the Hall of Science at the Century of Progress Exposition in Chicago in 1933 (see page 79). In this picture mathematics is represented as the main root of the tree, and springing from it are the other roots, stems, and branches representing the various basic and applied sciences. Copies of this painting can be secured at a nominal charge,[1] either in colors or black and white, and in a size suitable for framing to hang on the walls of mathematics classrooms or in a small size suitable for pasting in the front of individual textbooks. This affords an excellent means of keeping before the students a continual reminder of the contribution which mathematics makes to these other subjects and fields of study.

The dependence of physics, chemistry, and astronomy upon mathematics is so manifest that it is hardly necessary to dwell upon it here. Mathematics is literally indispensable in the study of these subjects, and no informed person would question its instrumental value in this connection. It has only been in comparatively recent years that biologists have begun to realize the vast possibilities arising out of the application of mathematics to their science, but the work of Quetelet, Galton, and Pearson and their followers has opened up new avenues of approach and innumerable possibilities for systematizing and expanding this science and for the investigation and interpretation of biological phenomena through the formulation of precise mathematical expressions of the relationships and changes involved. Remarkable advances have been made through the application of mathematical procedures to advanced studies in genetics, heredity, nutrition, growth and maturation, senescence, metabolism, fatigue, the effects of various stimuli on organisms, and many other special phases of biological and physiologi-

[1] Inquiries and orders should be mailed to the Business Manager of the Museum of Science and Industry, Jackson Park, Chicago.

cal study. Indeed, it is hardly too much to say that it is no longer possible to pursue the study of biological phenomena very far beyond the early descriptive stages without the aid of mathematical analysis and treatment.[1]

The social sciences are also beginning to draw heavily upon mathematics, particularly statistical and graphic methods, for the investigation and interpretation of social phenomena. Much of the mathematics used in connection with these subjects is so simple and so enlightening that it can be incorporated easily and very appropriately even in junior-high-school courses in which are studied such matters as public health, safety campaigns, thrift, population trends, expenditure of public moneys, and other topics which deal in simple fashion with social and economic phenomena. Economics and sociology deal essentially with mass phenomena, and there is a widespread feeling that the only mathematics which is used in connection with these subjects is statistics. It is quite true that statistics, including graphics, is more extensively and more obviously used than other mathematical procedures in the elementary work in these fields, but in some of the more advanced work, especially in economic theory, there are to be found important applications of mathematics which are nonstatistical in nature.[2]

In one way or another mathematics leaves its imprint upon the foundations of many of the school subjects. Its applications are more manifest in some than in others, but seldom indeed, if ever, are they lacking altogether. We have seen that they are not limited to the physical sciences but have important bearings upon the biological and social sciences as well. The industrial arts require mathematics. Psychology is finding more uses for it all the time. Even English, the

[1] J. Arthur Harris, The Fundamental Mathematical Requirements of Biology, *The American Mathematical Monthly*, **36** (1929), 179-198.

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Nathan H. Woodruff, The Mathematics Used in Biology, *Contribution to Education* 267 (Nashville, Tenn.: George Peabody College for Teachers, 1940).

[2] Commission on Post-War Plans, Guidance Report, *The Mathematics Teacher*, **40** (1947), 315-339.

Douglas E. Scates, Statistics—The Mathematics for Social Problems, *ibid.*, **36** (1943), 68-78.

foreign languages, and the fine arts are enriched by an understanding of the mathematical principles of form and number, of symmetry and order, upon which they are based. By continually impressing upon the students these relationships and applications of mathematics to other school subjects, teachers can stimulate interest in the study of mathematics and can at the same time give the students a more comprehensive and complete idea of the nature of these other subjects.

Motivation through Showing the Application of Mathematics to Business, Industry, and the Professional Fields.[1] A very important means of stimulating interest in mathematics is through pointing out its applications to fields of work through which people gain their livelihood. Students are interested in this not only from an academic standpoint but for practical reasons as well. All boys and many girls must look forward to the intensely practical problem of selecting an occupation and earning a living, and they are generally interested in learning something about the opportunities and requirements in different fields. The extent to which mathematics enters into the upper levels of many lines of work is not realized by most people. By pointing out these applications, teachers can perform valuable service in the way of guidance to the students and at the same time stimulate their interest in mathematics itself.

It should be emphasized that professional work in a number of fields requires extensive training in mathematics and, indeed, its continual use. In a preceding section the application of mathematics to the natural and social sciences has been discussed, and it is obvious that individuals desiring to take up professional work in teaching or research in these sciences will be either entirely prevented from doing so or will be greatly handicapped in their work unless their interest and aptitude for such work is instrumented by the ability to adapt mathematical techniques to the circumstances and problems peculiar to their particular lines of study. In this connection the attention of students should be called to the fact that mathematics is now coming to be recognized as a necessary part of the professional equipment in such fields as anatomy, physiology, psychology, psychiatry, and medicine.

Students scarcely need to be told that mathematics is the foundation of engineering. While mechanical devices such as the integraph and handbooks of tables and formulas have reduced the need for the actual applications of calculus and differential equations in much routine engineering work, situations will inevitably arise from time to time to which no ready-made formula will apply. In such cases the engineer

[1] Commission on Post-War Plans, *loc. cit.*

needs to be able to size up the situation, to detect the principles and relationships involved, and to express and investigate these by means of mathematics. A good working knowledge of college algebra, trigonometry, analytic and descriptive geometry, calculus, and a considerable familiarity with differential equations is generally regarded as essential for carrying on or directing engineering work. The knowledge that these courses are almost invariably prescribed in engineering courses has served as an effective spur to many students.

Industry and commerce make somewhat less extensive demands upon the individual for mathematical training than do the fields which have just been discussed. It will stimulate the students' sense of the usefulness of mathematics, however, even in the field of business, if they are impressed with the fact that executive positions generally require the ability to gather, organize, analyze, and interpret complex statistical data and that the businessman who has been trained in statistical methods and in the mathematics of business and finance has a notable advantage over the one who lacks such training. It should be made clear to the students that such training increases one's chances of promotion to the more important and remunerative positions if he is in the employ of a large corporate business and that, if he owns and operates his own business, such training can be valuable to him by enabling him to appraise his operations thoroughly and to plan them upon the basis of factual information rather than upon guesswork.

Motivation through Emphasis on Cultural and Educational Values. While the practical motive for the study of mathematics is a powerful one, teachers should not neglect to point out its cultural and general educational values. It should be the responsibility of the teachers to emphasize continually that it is an essential part of culture and education to understand the background and the nature of the developments which are going on in the world. It is no easy task, however, to keep oneself informed in these matters in a world in which social, economic, and especially scientific changes are taking place with the rapidity characteristic of the present time. Students should be impressed with the fact that many of these important developments which directly affect our daily lives cannot be adequately understood except through an understanding of scientific principles whose development, expression, and interpretation depend, in turn, upon mathematical principles. They should be led to see that mathematics will aid them even in such matters as interpreting social and economic phenomena and that it is indispensable to the understanding and development of scientific theory.

Students will sometimes argue that it is not necessary to understand

these things; that one can operate an automobile effectively without knowing the adiabatic formula or can turn on a radio and enjoy the program without knowing even of the existence of Maxwell's equations. All of which, of course, is true. The thing which students so often fail to realize, however, and which should be impressed upon them continually is that to limit their interest in such things to the bare utilitarian aspect is to miss the real thrill and wonder of them as well as to run the risk of setting up a barrier to subsequent intelligent study of their characteristics. It is not necessary to be able to solve difficult differential equations in order to be able to *appreciate* the role which mathematics plays in the development of modern science, but such appreciation can hardly be attained without some considerable study of mathematics beyond that which is needed to compute the area of a field or to keep oneself from being shortchanged. Indeed without such a background it is impossible to read understandingly not only a great many scientific articles of general interest but also a rather large number which are written for popular consumption.

Mathematics provides an outlook, and a means of understanding. There are important aspects of the world which only mathematics can interpret to the citizen. Mathematics affords one a mode of thinking about many aspects of life, and a very general kind of language. A liberal view of education regards such matters as genuine needs of even the ordinary citizen. It rejects the thought of being satisfied with such a minimum working equipment as would enable him to exist as nothing more than "a hewer of wood and a drawer of water."[1]

Aside from the technical aspects of the subject the *postulational method* of mathematics has a major contribution to make to the cultural education of the individual. If students are kept aware of the nature and the universal applicability of this method, it will be found to provide an exceedingly strong motive for the study of mathematics, not only from the standpoint of its cultural significance but also because of its intrinsic interest. An appreciation of the significance of the "if-then" type of reasoning is one of the most important potential educational and cultural values of the study of mathematics. When once attained, it in turn not only makes mathematics infinitely more meaningful but infuses a keener interest into the study and provides

[1] Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, *Fifteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1940), p. 210.

one of the most powerful motives for the continued pursuit of work in this field.

Motivation through Mathematics Clubs and Recreations. The concept of motivation has become identified in the minds of a large number of people with the idea of games, puzzles, plays, anecdotes, and other interesting, but sometimes more or less trivial and unrelated, matters often referred to under the generic title of mathematical recreations. It is unfortunate that this should be the case, because motivation implies a much broader and more significant connotation than this and takes place through various avenues. At the same time such mathematical recreations are valuable and legitimate in relieving the tedium of necessary routine work and in presenting an aspect of mathematics the existence of which is at times not even suspected. It is a rare individual, especially child, who is not interested in games or in things which are unusual or unsuspected and which contain elements of surprise or of mystery. While mathematical puzzles, contests, and games cannot be permitted to pre-empt too much of the time allotted to regular classwork, there is abundant evidence that the moderate and appropriate employment of such devices does add much of interest and zest to the courses, especially in the junior high school.

Mathematics clubs provide an excellent means of stimulating and fostering mathematical study. Membership in these clubs is usually voluntary, and for this reason the clubs are composed mainly of students who have a real interest in mathematics and who desire to obtain a view of the subject which is somewhat different from that gained in the classroom. Such clubs offer excellent opportunities for free consideration of matters of special interest to the members without the necessity of having the programs follow any particular organic sequence of topics such as is generally necessary in regular class instruction. Secondary-school pupils, like all others, are dependent upon each other in their mental, physical, social, domestic, and other relationships. They listen to ideas expressed by others and add their own; they criticize and are criticized. The fact that they do not always agree stimulates interest and motivates discussion. A mathematics club offers an ideal place for a free exchange of mathematical ideas and for frank and helpful criticism of these ideas. The club also makes possible an informality and a social atmosphere which the classroom can hardly provide. The club should be an organization of, by, and for the students, the teacher being a sympathetic counselor whose main function is to foster a continuance of interest and to cooperate in guiding the activities of the club along appropriate lines.

The principles of organization of a mathematics club should be

neither numerous nor involved. The objectives should be clearly stated and well understood by all members. A *sine qua non* of effective organization should be an emphasis on pupil activity. There should be a faculty advisor who should be inconspicuous but ready to function when needed. In general, any criticisms that he might have to make concerning programs should be given in private to the individual pupil or pupils concerned. It is desirable that each club limit its membership to such size that participation by all will be possible. All meetings should be held at regularly scheduled times, and there should be maintained a balance between the purely social, the purely informational, and the mixed social and informational type of program.

The programs of mathematics clubs may cover a wide range of topics, many of which have been listed and discussed in numerous books and articles. These will include topics drawn from the history of mathematics, including biographical sketches and interesting anecdotes, the evolution and development of certain aspects of present-day mathematics, topics from algebra, geometry, arithmetic, or trigonometry, games and contests, and applications of mathematics to other subjects and fields of activity. The nature of the programs and topics to be discussed will of necessity depend considerably upon the age and advancement of the members of the club. Some subjects which could be discussed with interest and profit by students in the junior college or in the upper years of the senior high school would not be appropriate for junior-high-school clubs, whereas certain activities in which these clubs could well engage would be too elementary to hold much interest for the older students. The membership of the mathematics club should be so far as possible fairly homogeneous as regards age and grade level in order that programs may be arranged which will be of interest to all the members.

In his book entitled "The Teaching of Mathematics" Schorling[1] has presented a valuable list of suggestions and illustrative materials for mathematical recreations. Another good source is the book "The Teaching of Junior High School Mathematics" by Smith and Reeve.[2] Other helpful sources, most of them more recent than those mentioned above, are given here for convenient reference.

BRESLICH, E. R.: "The Technique of Teaching Secondary School Mathematics" (Chicago: University of Chicago Press, 1930), pp. 63–65, 69–71, 72, 75, 77–78, 82, 85–86.

[1] Raleigh Schorling, "The Teaching of Mathematics" (Ann Arbor, Mich.: Ann Arbor Press, 1936), pp. 224–240.

[2] D. E. Smith and W. D. Reeve, "The Teaching of Junior High School Mathematics" (Boston: Ginn & Company, 1927), pp. 359–403.

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In addition to these bibliographies it will be worth while to note some continuing sources of ideas for motivation in mathematics teaching. These appear currently as regular departments in certain journals. Among them the following ones can be very helpful:

Clubs and Allied Activities, *American Mathematical Monthly*. (In each issue can be found accounts of programs and activities of mathematics clubs in various colleges.)

Elementary Problems and Solutions, *American Mathematical Monthly*. (Special problems and solutions appear in every issue. These problems presuppose two years of college mathematics.)

Curiosa, *Scripta Mathematica*. (Descriptive accounts of mathematical curiosities appear in every issue.)

Recreational Mathematics, *Scripta Mathematica*. (Descriptive accounts of mathematical games, puzzles, and other recreations appear in every issue.)

The Problem Corner, *The Pentagon*. (Each issue contains a section given over to special problems and solutions.)

The Mathematical Scrapbook, *The Pentagon.* (Each issue contains a section made up of mathematical curiosa and miscellanea.)

Topics for Chapter Programs, *The Pentagon.* (Each issue contains suggestions of topics for programs of mathematics clubs, and also topical bibliographies.)

Problem Department, *School Science and Mathematics.* (Special problems and solutions appear in every issue.)

Notes from a Mathematics Classroom, *School Science and Mathematics.* (Suggestions on specific teaching problems appear in nearly every issue.)

Problems and Questions, *Mathematics Magazine* (formerly *National Mathematics Magazine*). (Special problems, questions, and solutions appear in nearly every issue.)

The Art of Teaching, *The Mathematics Teacher.* (Most issues contain descriptive and specific suggestions of teaching procedures referred to particular classroom problems.)

Aids to Teaching, *The Mathematics Teacher.* (Descriptions, discussions, and appraisal of various teaching aids for mathematics appear in every issue.)

Miscellanea—Mathematical, Historical, Pedagogical, *The Mathematics Teacher.* (A "swapping post" for all kinds of notes which add interest to mathematics teaching, this department is of recent origin but probably will appear henceforth in every issue.)

The following list brings together at this point a considerable number of detailed references which will be found helpful in locating materials dealing with mathematical clubs, programs, and recreations. The books include some which were listed in the original edition of this book. The periodical references, on the other hand, include very few which were published before 1940. For earlier references, the reader should consult the list in the original edition of this book or some of the other lists given above.

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The Use of Multisensory Aids. In their efforts to put increased meaning and interest into work in mathematics, teachers are drawing more and more heavily upon devices which have direct sensory appeal and which at the same time exhibit or clarify mathematical concepts and relationships. Pictures, posters, drawings, charts, models, instruments, slides, silent and sound motion pictures, demonstrations, projects, exhibits, club programs, and even the radio are examples of such devices. The proper use of such teaching aids has complete psychological justification, and the experience of the armed forces during World War II in the use of concrete instructional aids provides indubitable evidence of their effectiveness. Ideas are almost invariably cleared and strengthened by concrete illustration or demonstration, and interest usually accompanies understanding. There can be no doubt that the well-considered use of good and appropriate multisensory aids in teaching mathematics will pay good educational dividends.

Of course, the use of sensory aids in teaching is not a new thing. Indeed all teaching has always involved the communication of ideas through the senses either orally through the medium of speech or

visually by use of written or printed material. Textbooks, writing materials, and the blackboard, all of which are sensory aids, have long been regarded as indispensable equipment for mathematics classes. Nor has the use of sensory aids in teaching mathematics been limited to verbalization. For many years resourceful teachers have used models, instruments, drawings, and other devices to stimulate interest and facilitate learning. Augmented by the development of new means for auditory and visual presentation and by a sounder appreciation of how children learn, there has been over the past half century a gradually widening interest in this aspect of teaching. But for a long time the potential values of these supplementary devices were fully realized only by exceptional teachers, and only recently has there been a concerted effort among leaders in mathematical education to alert all teachers to these possibilities.

The most comprehensive and significant step in this direction has been the publication in 1945, by the National Council of Teachers of Mathematics, of its *Eighteenth Yearbook* entitled Multi-Sensory Aids in the Teaching of Mathematics. This book was the culmination of a movement which began to take definite form at a meeting of the National Council in 1937 and which was carried on in subsequent years under the stimulation and direction of the Council's Visual Aids Committee and, later, its Committee on Multi-Sensory Aids. It is a book of real professional significance to teachers of mathematics, owing to the fact that it contains many concrete suggestions for practical classroom use. Furthermore, its two appendices contain short descriptions and excellent illustrations of individual models and teaching devices, an annotated bibliography of about 500 books and periodicals, and a list of available films and film strips. This *Yearbook* is one of the best available sources of ideas and helps for the construction and use of multisensory aids in the teaching of mathematics. It has probably contributed materially to a growing interest in the use of multisensory aids not only among classroom teachers but also among producers and manufacturers of instructional aids.

Evidence that increasing attention is being given to this development can be found in the contents of recent professional journals; in the programs of clubs and associations of mathematics teachers; in the programs and activities of mathematics workshops and institutes; and in the accelerated activities of commercial manufacturers of instruments, mechanical aids, models, classroom and field equipment, slides, films, projection equipment, and other audiovisual aids. It can also be found in the growing advocacy for the inclusion of training in such

laboratory methods and field work in the collegiate programs of those who are preparing to teach mathematics.

> . . . The future teachers of mathematics will make use of sensory aids in their own study of college mathematics. They must also learn the proper use of such aids in the teaching of high school mathematics. . . . We can expect that many teachers will continue their education while in service, and for these we can create courses in audio-visual education. But to some extent, we must provide an adequate coverage of methods of sensory education in the four year [college] program. . . .
>
> There remains the problem of additional training in the use of multisensory aids in the high school instruction. These aids and the theory of value and place in the curriculum could be a part of the "methods course." They can also be treated in part in courses on the history of mathematics. . . . [but] Does not the ideal use of multisensory aids in the teaching of mathematics lie in the mathematics laboratory? In the laboratory will be found all the measuring instruments, as well as material for constructing simple instruments; calculating devices of all sorts; geometrical models both fixed and with movable parts; models showing relations under variance, and also invariant properties; surveying and astronomical instruments of all kinds; drawing boards and drawing instruments; charts and globes; and any additional devices that can be used to exemplify mathematical principles.[1]

It must be admitted that there are limitations and handicaps to the use of multisensory aids in mathematics classes. It should never be forgotten that they are *aids* to teaching, not substitutes for teaching. Few if any material aids are completely self-teaching. Moreover, unless careful planning precedes activities which involve the use of instruments or material aids, and unless the plan envisions for these aids very definite and particular mathematical contributions, their use might easily degenerate into just a sort of entertainment. Whatever topic is under consideration and whatever devices are used, the teacher himself must bear the responsibility for seeing that the mathematical objectives are attained.

In addition to this pedagogical limitation there are certain very real practical handicaps which cannot be disregarded or simply wished away. One study, recently reported, lists the main obstacles to the use of multisensory aids in mathematics as lack of time, lack of money, lack of information about teaching aids, and lack of facilities for using some teaching aids (especially films and aids requiring projection

[1] Howard F. Fehr, The Place of Multisensory Aids in the Teacher Training Program, *The Mathematics Teacher*, 40 (1947), 212–216.

equipment).[1] These are all understandable, and it is not to be expected that these handicaps can be completely removed. Some very desirable equipment and facilities will continue to be expensive, and it is to be feared that heavy routine teaching loads will continue to make correspondingly heavy demands upon the time of teachers. But alert teachers can at least become well informed about multisensory teaching aids and their uses through their professional reading and by attendance at meetings where these things are discussed, exhibited, and demonstrated. Once they become aware of how many teaching aids can be made or collected at no cost or else can be purchased at little expense, they can see that it is possible to provide for themselves many useful devices to make their work more interesting and more effective than it might otherwise be. The following list of selected references, most of which have been published in the past ten years, is offered in the hope that it may provide some immediate help in this direction.

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A Final Word. The means and devices which have been discussed in the foregoing pages will be found helpful in stimulating and maintaining interest in mathematics. In themselves, however, they cannot be regarded as panaceas or guarantees. In the last analysis the first

and greatest factor in creating interest is a sympathetic, well-informed, competent, and inspiring teacher. Not all the devices in the world can bear the fruit of a continuing and enthusiastic student interest if they are grafted upon the dead stump of instruction in the hands of an incompetent or uninterested teacher. The truly inspiring teacher must first of all be thoroughly grounded in the subject matter of mathematics, well beyond the level of any material which he is expected to teach, in order that he may inspire the confidence and respect of his students. He must have a sympathetic understanding of their difficulties and must be always ready and willing to offer proper guidance and stimulation.

Finally he must have an enthusiastic interest in his subject and in teaching it. He must believe in its values and its contribution to the educational well-being of the students. Enthusiasm is contagious, and sane enthusiasm backed by sympathetic and enlightened competence is the only real guarantee of the effective maintenance of student interest. Devices are helpful but they are not sufficient unto the task. The inspiring teacher is the real *sine qua non.*

Exercises

1. Construct a piece of demonstration apparatus to illustrate the principle that all angles inscribed in a given segment of a circle are equal.

2. Assume that you have been asked to present a 20-minute talk to a ninth-grade mathematics club. Select an appropriate topic, outline your talk, and be prepared to give it orally.

3. Present an annotated bibliography of references on mathematics clubs.

4. Do you think that a more extensive use of mathematical contests within or between classes or schools would serve to increase interest in mathematics? How would you plan for such a contest?

5. Make a detailed plan for carrying on a selected field problem with a class of 30 seventh-grade students. Submit this plan for criticism.

6. Describe several exercises involving field work which could be used with a class in junior or senior high school and which you think would increase the interest of the students in their work in mathematics.

7. Prepare an outline and summary of the Guidance Pamphlet published by the National Council of Teachers of Mathematics. Explain in some detail how this pamphlet could be useful as a means of motivation.

8. Much is being said these days about using motion pictures, film strips, and other multisensory aids as means of stimulating interest in mathematics. Discuss the advantages and also the limitations involved in the use of these aids.

9. Consult the *Nineteenth Yearbook* of the National Council of Teachers of Mathematics, and prepare a talk on the development and use of surveying instruments.

10. Most students are keenly interested in puzzle-type problems even though these may have no apparent applications. Do you think this would justify the

CHAPTER VII

MEANS TO EFFECTIVE INSTRUCTION

Effective instruction cannot be guaranteed by any single simple formula. It goes without saying that, if instruction is to be really effective, the subject matter must be selected and organized in such a way as to make it appropriate and suited to the age and intellectual development of the students. Further than this, it must be presented in an understandable and interesting way, and there must be provision for ample practice. Skills and concepts once developed must be maintained through reapplication and not allowed to deteriorate through disuse. Since students do not learn with equal facility nor at equal rates, there must be provision for individual differences. If the instruction is to attain a maximum of usefulness, it must be carried on with the deliberate purpose of securing a maximum of transfer and in such a way that the relation of mathematics to other fields of learning and activity is made manifest. These considerations involve careful planning and adequate testing of outcomes.

Three Fundamental Problems of Instruction in Mathematics. Mathematics is a cumulative and a continuously expanding subject both in its theory and its applications. At every stage the teacher of mathematics is confronted with three basic problems, *viz.*, (1) helping the students to develop understanding and mastery of new concepts, principles, relationships, and skills; (2) helping them to maintain understandings and skills already attained; and (3) helping them to secure maximum transfer of learning to their social environment. These three phases of teaching should be interwoven as far as possible into a unified instructional program, but their implications are essentially distinct and supplemental rather than identical. The teaching of new material necessarily draws upon the already established background as a frame of reference, and to this extent serves as a means of maintenance, but such maintenance is relatively incidental to the mastery of the new material and must be so regarded. Adequate maintenance and maximum transfer, especially of skills, cannot be assured by incidental contacts but require an instructional program designed especially for their attainment.

TEACHING FOR UNDERSTANDING: DEVELOPMENTAL TEACHING

As generally conceived, the foremost problem of direct instruction in secondary-school mathematics is the teaching of new material. It is this phase of instruction that makes the heaviest demand upon the skill and artistry of the teacher. The primary jobs are to explain, to make clear, to challenge, to guide to discovery, to develop understanding. In order to meet these responsibilities, the teacher not only needs to consider the logical relationships involved in the unit or topic but must also be keenly aware of the relation of the new concepts to the experiential background of the students. He must also be able to anticipate probable difficulties and to detect and clear up actual difficulties as they occur in the course of the development. He must be able and willing continually to view the unfolding and (to the students) unfamiliar subject matter not merely through his own experienced eyes but from the standpoint of immature students to whom it is all new and strange.

Inventory and Preview. Effective instruction requires that time shall not be wasted and that interest shall be stimulated and conserved. When new work is taken up with a class the assumption is generally tacitly made that the students are all entirely unfamiliar with the new materials which they are to study, but this is not always the case. It not infrequently happens, especially in some of the junior-high-school work, that the students will already have acquired some acquaintance with certain parts of the material presented in the textbooks. In some cases they will be found to have a fair degree of mastery of the supposedly new concepts and procedures. It is both wasteful of time and deleterious to the maintenance of interest to go through the motions of teaching students things with which they are already familiar. They become restless and impatient. Attention wanders, disciplinary difficulties are likely to be created, and the whole atmosphere of the class situation is rendered unfavorable to effective learning.

Therefore, before planning the work of a new course or a new unit, it is desirable that the teacher find out in the beginning as much as he can about the students' background with reference to such abilities and information as will be required in the new work. This may be done sometimes by means of oral questioning and discussion, while in other cases suitable written inventory tests will be more satisfactory.[1] Such tests are discussed and illustrated in Chap. VIII. By means of a

[1] E. R. Breslich, "The Technique of Teaching Secondary-school Mathematics" (Chicago: University of Chicago Press, 1930), pp. 4–11.

preliminary investigation of this nature the teacher will be in a position to proceed intelligently in building up necessary backgrounds and in presenting the new work to the class in an effective manner.

In presenting a new unit of work, it is important that the students be given at the outset a preview of the unit as a whole in order that they may get a perspective of the major ideas, issues, and principles in the unit and of the relations of these to each other, to the unit as a whole, and to the previous parts of the work. Not only does a preview of this sort give meaning and relevance to the larger ideas of the unit, but it gives significance and motive to the detailed study which must make up a major part of the students' work in the mastery of the unit. Teachers of mathematics have generally failed to recognize the importance of the preview as a part of instructional procedure. All too often new work is taken up detail by detail, just as it appears in the textbook, with little thought of relating the details to the structure of the unit as a whole. It is safe to say that this is partly responsible for much of the current dissatisfaction with the mathematics courses in our schools.

The preview should generally be given in the form of a well-organized talk by the teacher. It should not be a long talk because the span of concentrated attention for most students of secondary-school age is not great. Moreover, the purpose for which it is given does not require a lengthy discussion and would, in fact, be defeated by too much attention to detail since this would tend to obscure the larger principles. The teacher should present the ideas that he wishes to emphasize in a consecutive account in which clear concise statements would be embellished by supplementary discussion only insofar as it will aid in making the statements definite and understandable to the students. During this preview the students should be mainly in the role of listeners. Interruptions and questions should not be prohibited, of course, but for the most part students should be urged to defer their questions until the conclusion of the teacher's discussion in order to avoid interrupting the continuity of thought, which is a most important feature of this phase of instruction. The preview offers great possibilities for stimulating the curiosity and interest of the students and for helping them to cultivate the art of being good listeners. It should generally be followed by a brief test to determine how effective the presentation has been and whether there is need for rediscussion of the outline before the class proceeds to a detailed study of the unit.

General Methods of Teaching New Material: The Lecture Method. No mere lecture procedure such as is commonly used in college classes will suffice for the job of teaching secondary-school mathematics. It

is a common fault of teachers to employ too extensively the method of "telling" or of giving a coherent discussion of a topic and then to proceed as if on the assumption that the discussion has been completely understood and followed by the students. This assumption, however, is almost never justified. Secondary-school students are seldom able to assimilate adequately and immediately any lengthy one-sided teacher-given discussion of unfamiliar subject matter. Points of difficulty will inevitably arise, and, unless these are cleared up promptly, they will fail to register with the students; failure to get these points cleared up may easily result in blocking the understanding of the subsequent parts of the discussion.

This does not mean that "telling" is always and entirely out of place. On the contrary, there are many times when judicious telling or explanation may be not only proper and valuable but absolutely necessary, as, for example, in making clear the meaning of new terms and concepts. Such use of the "telling" or lecture method in secondary-school mathematics, however, should generally take the form of explanations or illustrations, and these should not be protracted longer than necessary. Moreover, the discussion should not be one-sided. It should be interspersed with frequent questions by the teacher who should also strive to elicit questions and contributions from the class.

It is not always easy to get students to raise questions because all too often the difficulties which arise in their minds are not well enough defined to enable the students to put them into words. Many students are quite sensitive about appearing slow of perception or "dumb" in the eyes of their classmates, and, rather than run the risk of embarrassment, they commonly let these matters pass in silence. For no better reasons than these, students will frequently allow statements to go unchallenged even though they do not understand them. Such barriers to freedom of inquiry on the part of the students can be broken down only by tact and sympathetic encouragement. The teacher should never resent an interruption of discussion when the interruption is caused by the raising of a legitimate question or inquiry. On the contrary, students should be given every encouragement to raise such questions at any time when they are unable to follow the discussion clearly.

At best, however, students cannot be depended upon to bring up all points which may need special attention in the course of a discussion or explanation. The teacher must anticipate these as far as possible and be always alert to detect them as they become apparent. This can often be done by noting the puzzled expressions on the faces of the

students even though they may not actually raise questions. Always at such times, and frequently in any case, the teacher should check the understanding of the discussion by means of questions addressed to members of the class, and at the completion of the discussion of any new topic a check test of some sort should be given before passing on to other activities. Merely to present a finished discussion, closing with some such general question as "Is that clear to all of you?" or "Are there any questions?" is entirely inadequate. The silence with which such a question is generally received is absolutely no assurance that the class has followed the discussion at all, though as a rule this interpretation is wrongfully placed upon it. It may mean simply that the students have not followed the discussion well enough even to be able to ask intelligent questions about it.

The Heuristic Method. In contrast to the lecture method there are certain other ways of presenting new material which aim to avoid the shortcomings of the lecture method. These have been discussed adequately in other books,[1] but for the convenience of the reader they will be summarized briefly here.

The *heuristic method* of teaching is the antithesis of the lecture method. It is a method which aims to lead the student, by well-chosen questions, to discover facts and information, relationships, and principles for himself rather than having them handed out to him in the manner of direct information by the teacher. To use this method effectively, the teacher must be very skillful in the art of questioning and must be adept at sensing precise key points of difficulty which perhaps are not definitely recognized even by the student himself.

The advantages of the heuristic method in teaching mathematics, where it is successfully employed, are manifest. It makes the student an active participant in the learning process and provides a spur to quicken his interest since it places him in the role of at least a quasi investigator rather than a mere passive recipient of information. The fact that the discoveries which he makes have been made previously by someone else neither alters nor detracts from the fact that to his mind they are new and largely original. That he has been guided toward

[1] See any of the following references;

Ibid., pp. 29-37.

J. O. Hassler and R. R. Smith, "The Teaching of Secondary Mathematics," (New York: The Macmillan Company, 1930), pp. 140 170.

Arthur Schultze, "The Teaching of Mathematics in Secondary Schools," (New York: The Macmillan Company, 1927), pp. 30 19.

J. W. A. Young, "The Teaching of Mathematics" (New York: Longmans, Green & Co., Inc., 1924), pp. 53–151.

his discoveries by the helpful and stimulating questioning of the teacher should not detract from his justifiable pride in his achievement. The student's part as an active participant in the unfolding of a portion of the mathematical scroll seldom fails to add zest to his work and to give him a more complete and enduring mastery of what he has learned.

On the other hand the heuristic method, essentially an individual method, is undeniably slower than the lecture method, especially in the earlier stages of the work, and it is much more difficult to use. It will be effective only in the hands of a teacher who has great patience together with a high degree of insight into the workings of the student mind and of skill in the use of the question for the purpose of accomplishing certain desired results. If students are to be helped in discovering things for themselves, the questions must not be allowed to degenerate into a mere true-false type in which the nature of the answer is so evidently implied that the element of discovery is largely removed from the situation, leaving little to the imagination of the student.[1]

The Genetic Method. Skillful use of the heuristic method tends to develop an attitude of mind which is most favorable to successful work in mathematics. A variation of this procedure (sometimes called the *genetic method*) aims to retain its spirit and advantages and at the same time to remove some of its limitations by having the questions directed to the entire class or group instead of merely to one individual. Thus the intention is that the class will be guided toward discoveries as a cooperating group rather than as separate individuals. In the hands of a trained and skillful teacher this method may be expected to achieve excellent results, and in such hands it is perhaps better adapted in general to the successful development of new material with classes than is any other single procedure. It is, in fact, a combination of several procedures: questioning; giving information as and when needed; eliciting student participation; explaining, guiding, illustrating, stimulating, and evoking interest and curiosity; and continual checking of the understandings and the reactions of the students. These activities keep the teacher closely in contact with the entire class situation and keep the members of the class in contact with each other and with the teacher at all times. It is probably the most difficult of all methods of teaching because it cannot be stereotyped; the teacher, in addition to possessing a broad and deep competence with regard to the subject matter, must be not only a technician but also a skilled artist at the job of teaching. In the hands of an unskilled teacher this

[1] Schultze, *op. cit.*, p. 46.
Young, *op. cit.*, p. 72.

method is likely to degenerate into random discussion from which little will be gained and in the course of which much time may be wasted. On the other hand, a trained and skillful teacher can achieve excellent results through the use of such a procedure.

The Laboratory Method. The laboratory method is another procedure for stimulating activity and discovery on the part of the students and for avoiding the disadvantages of the lecture method. As the name implies, the idea underlying this method of teaching is that students will develop new concepts and understandings particularly well through experimental activities dealing with concrete situations such as measuring and drawing, weighing, counting, averaging, estimating, taking readings, comparing, analyzing, classifying and checking data, and deriving original quantitative data from concrete physical situations. Most work of this nature will involve the use of various kinds of physical and mathematical instruments, especially measuring instruments. Much of the work will be done in the classroom, but some may take the form of elementary outdoor field work such as making simple surveys and doing elementary mapping of small areas, laying out tennis courts or baseball diamonds, or measuring heights and distances indirectly by the use of simple field instruments such as were described in an earlier chapter. Proponents of the laboratory method maintain that students attain better mastery of mathematical concepts and principles by deriving them in this way from concrete experience and that these concepts and principles become more functional and meaningful when they are seen in relation to actual applications.

There is undoubtedly much validity to this argument. On the other hand, mathematics per se is neither a physical nor an experimental science in the sense that the natural sciences are, and, while well-conducted laboratory work of the nature indicated can do much to supplement and enrich some parts of mathematical study, especially in the lower grades and the junior high school, it can never provide a complete foundation for mathematical work. Mathematics is a distinct field of study in its own right. It is true that it has innumerable points of contact with other fields, but it also has characteristics peculiar to itself. Experiments and applications are valuable in relating mathematical principles and processes to other fields, but they do little toward developing and clarifying the interrelations of the different parts of mathematics itself. They can be made to provide valuable supplementary work for purposes of enrichment of certain parts of mathematics, but beyond this, too much may not be expected. Furthermore, laboratory work in mathematics may easily degenerate into

more or less aimless playing with instruments unless it is carefully planned, supervised, and guided toward definite ends and unless adequate equipment is available. Experimentation which is mere busywork and which does nothing to develop understandings of principles or applications is practically worthless. The responsibility for the effectiveness of laboratory work in mathematics lies squarely upon the teacher.

Developmental Teaching. In discussing the advantages and disadvantages of various general methods of teaching new material, the intention has been to emphasize the fact that there is no single method which fits all situations. In order to develop new material successfully, the teacher must adapt his procedure to the situation as he finds it and must modify his procedure in accordance with the changing requirements of the situation. Developmental teaching is an art. It can be neither standardized nor stereotyped. Procedures which are used successfully by one teacher may prove to be unsuccessful when tried by another, or perhaps even by the same teacher under different circumstances. Much depends upon the personality of the teacher, upon his enthusiasm, tact, understanding of children, and upon his ability to sense intuitively the procedures which will serve best to capitalize the psychological classroom situation of the moment or to modify it in such a way that it may be made to contribute most powerfully in the drive toward the objective which has been set. That teacher will be most successful in developmental work who has at his command *various* methods of procedure and who uses them in such a way as to make them supplement each other most advantageously as occasion may indicate.

One of the greatest mistakes which many teachers make is to try to cover too much ground in a given period of time or to try to cover a given amount of material in too short a period of time. This nearly always results in superficial learning or in no learning at all. Particularly in the developmental teaching of new material the teacher should avoid forcing the process too rapidly. The development of new concepts and principles is a slow process, and it always requires a certain amount of discussion. Sometimes it will be necessary for the teacher to carry the burden of the discussion in building up and coordinating the necessary background and in giving a first presentation of the new material, but this should be done in such a way as to avoid "lecturing" so far as possible. Where it is feasible to guide students into exploratory activities through appropriate questioning or laboratory exercises so that they may discover things for themselves, this should be done.

When it is necessary for the teacher to give direct information, it should be given briefly and concisely and should be checked by pointed and searching questioning. New understandings as they are developed should be given permanence, clarity, and interest by means of adequate illustration and application.

Developmental work is not the job of the teacher alone. In order to be successful, it requires the continuous interaction of the students' best efforts with those of the teacher. The aim at all times is to develop in the students a broadening background of mathematical understanding and to foster a continuing interest in the subject, to the end that the students will gain added appreciation of its nature and usefulness and will acquire increasing ability to do independent thinking in the field. The teacher must plan and direct the activities of the class toward these goals. He must strive to secure the highest possible degree of cooperative effort on the part of the students. He must be tactful and sympathetic, helping where necessary, encouraging, guiding, checking, and always stimulating the students to put forth their own best efforts. Such a program of developmental teaching may be expected to yield highly satisfactory results not only in developing mastery of the new subject matter immediately in hand but also in building up an added appreciation of mathematics and of its contributions, in developing an increasing ability to do independent mathematical thinking, and in stimulating interest in the pursuit of further mathematical study.

TEACHING FOR ASSIMILATION: DIRECTED STUDY

The preceding sections of this chapter have dealt with the *development* of new material, *i.e.*, with methods of presenting new material to the students, of discussing it with them, and of giving them their first basic understandings of it. It is well to reemphasize here that many of the difficulties which students encounter in mathematics can be traced to the inadequacy of the developmental work which precedes the period of independent study. Obviously students will not be able to make much independent progress toward the assimilation of concepts, principles, and relationships of which they have not even gained a basic understanding. Adequate developmental teaching is an absolute prerequisite to successful assimilative study in mathematics. The teacher who neglects to assure himself that his developmental work has been reasonably effective is likely to find his students groping helplessly for light on matters of which they have gained but an imperfect understanding. It is therefore of the utmost importance that, before the

students are set to independent study of new material, measures be taken to test their understanding of the ideas which the teacher has tried to develop with them as a preliminary basis for their work during the assimilation period. The results of such a test will often be extremely illuminating to the teacher in revealing points where further developmental work is needed. Any such points should be cleared up immediately by thorough reteaching.

This, however, is only one step toward real and adequate mastery. Concepts are not ultimately mastered without many illustrations in varied contexts, nor principles without repeated application, nor processes without extensive practice, nor any of these without protracted and sustained intellectual effort on the part of the students themselves. The purpose of developmental teaching is to give the students adequate bases of understanding and appreciation and motive upon which to build, but the process of mastery can by no means be thought of as ending with this step. On the contrary, this is merely a beginning which must be followed by an extensive period in which the student must devote himself to the task of assimilation and fixation of the ideas, principles, and processes which have been brought out in the developmental work. This is an indispensable part of the learning process.

The teacher has been described as having a very active and prominent part in the developmental work. In the assimilative stage, however, the situation is reversed. From this point on, the teacher's role, while no less important than in the developmental work, is much less obvious. Of course, there will need to be some general discussion from time to time for purposes of motivation and for clearing up points which offer persistent difficulties, but such discussions, instead of occupying the center of the stage, should be incidental to the directed study which is the characterizing activity on this part of the work. In the process of assimilation through directed study the students themselves are the main participants so far as overt activity is concerned. The task of the teacher now should be that of guiding and directing their work, stimulating them, encouraging them, helping them over hard spots, evaluating their progress, and in every way possible striving to get them to put forth their best efforts to achieve a permanent and functional mastery of the material upon which they are working.

Directed Study in Mathematics. Under the recitation and homework plan of teaching mathematics the students are often compelled to do their studying under conditions both physically and psychologically unfavorable to effective work. Many homes are not so arranged

as to make it possible for the children to have suitable desks in quiet rooms where they can study to advantage. The whole atmosphere of such homes is redolent of other things, and distracting elements are the rule rather than the exception. Most schools maintain some form of study hall during school hours. As a rule, these provide conditions which are more favorable for study than those which students find at home. They are at least normally quiet, and the fact that they are conducted during school hours eliminates certain psychological handicaps to effective study which are likely to operate outside of school hours. The well-conducted general study period cannot, however, ordinarily provide conditions which are as well adapted for most students to do really effective work as those made possible under a well-ordered plan of directed study. Nearly all students need help at times, but in many cases the teachers who are in charge of study periods have had so little training in mathematics that they are unable to be of any assistance in helping students over difficulties or in directing their work in this field. The one person who is best qualified to do this is the mathematics teacher himself. The time and the place where it can be done to best advantage is directly in the mathematics class. There the student can address himself completely to his mathematical work and can receive such assistance as he may need from the teacher who is in charge of this work and who is therefore in a better position than anyone else to give him the proper attention, assistance, and direction.

Suggestions for Conducting Directed Study in Mathematics. At first thought it may appear that directed study in mathematics makes little or no demand upon the teacher. It is true that the teacher's role is much less prominent than it is in developmental teaching, but it is hardly less important. Directed study does not reduce teaching to an entirely individual basis, but it does attempt to combine the main advantages of individual instruction for those students who need it with the economy of time and other advantages generally recognized as accruing to group instruction. Directing study involves much more than the maintenance of order. If it is to be really effective, the teacher must be continually in touch with the work of each individual student. This requires repeated inspection and quick sizing up of the difficulties and the needs of the various students. The teacher must be adept, not only at spotting key difficulties and in helping the students to clear them up, but also in discriminating between students who are experiencing real difficulties and those who are merely disinclined to work for themselves.

Many teachers make the mistake of rushing to the assistance of students at the first sign of difficulty and of virtually doing their work for them. This is bad practice for two reasons. In the first place it does the students little or no good. Mastery can come only through individual effort, and it is not likely to be gained by the student who is unwilling to assume substantial personal responsibility for results and who relies to an excessive extent upon the teacher's assistance. There will always be such students, and, if the directed study program is to be effective, the teacher must be able to detect them. He should decline to extend them assistance unless he is convinced that they are seriously in need of it, and he should make every effort to get them to modify their attitude so that they will be willing to take a larger share of responsibility upon themselves and to rely less upon assistance from the teacher.

In the second place, if the teacher allows students to impose unduly upon his willingness to help them, he will soon find himself so swamped with demands of this nature that he will be unable to take care of them all in a satisfactory manner. If he allows himself to be hurried to the point where he has to rush from student to student, not only will many students waste time waiting for him, but inevitably he will tend to give direct information instead of helping students to think their own way through their problems, and such ostensible assistance as he can give under such circumstances will largely defeat its own purpose. In addition to this, the pressure and stress will almost certainly leave him physically tired, mentally and emotionally disorganized, and generally unfitted for effective work. The only way in which these evil effects can be prevented is for the teacher to confine his assistance to those students who, in his opinion, really need it, and even there to the key difficulties which the students encounter. Perhaps merely the explanation of a word will be sufficient. It is extremely important that he be able to lead the student to disclose just what is causing his difficulty so that it may be cleared up without allowing extraneous matters to befog the issue and without unnecessary waste of time.

On the other hand, the teacher must not be niggardly with assistance where it is really needed. It is a fine art to determine just who really needs help, just what and how much assistance is needed, and in what manner it should be given. It can be done successfully only by a teacher who has a sympathetic understanding of the attitudes and abilities of his students, who has a good knowledge of the difficulties to be expected and of the errors commonly made in the work under consideration, and who possesses a trained insight which will enable him

to get directly at the root or key of the student's difficulty, even though the student himself may not know precisely what it is that is causing him trouble.

The following suggestions have been found helpful to teachers in conducting directed study in mathematics:[1]

1. Be sure that the preliminary developmental work has been clearly understood by the students before allowing them to begin their study. This work provides the basic foundation and framework with reference to which the subsequent assimilative study is oriented. That is, it provides the initial understandings which are to be assimilated, amplified, organized, and made permanent through subsequent study. Directed study can play its part in this process only if these preliminary understandings have been satisfactorily developed.

2. Be sure that the assignment of the work to be done is made clear to all the students, so that each one will know precisely what is expected of him.

3. Be sure that each student has the equipment which he will need for his work. The practice of borrowing breeds carelessness, wastes time, and causes disturbance. The teacher, however, should keep on hand a limited amount of equipment with which to provide against emergencies. If it becomes absolutely necessary for a student to borrow any equipment, it is better for him to borrow from the teacher than from a fellow student.

4. Soon after the directed study period has begun, make a rapid inspection of the work of all the students, noting which students seem to be most in need of help. Ordinarily it is best not to interrupt this survey for the purpose of answering questions or giving assistance to individuals. Such students as need assistance can receive it later. The purpose of the survey is to ascertain whether there is a need for immediate reteaching of any parts of the work and to see what students, if any, seem unable to get started with their work.

5. If the survey shows that a considerable part of the students are having trouble in getting started, it may be advisable to stop the study work and to do some general reteaching of any points which may appear not to have been developed clearly. If such reteaching is needed, it should be done immediately with the class as a whole. The study period can then be resumed.

6. If the survey shows that no *general* reteaching is needed, the

[1] The list presented here includes adaptations of the excellent suggestions given by Breslich in his book, "The Technique of Teaching Secondary-school Mathematics" (Chicago: University of Chicago Press, 1930), pp. 41–44.

teacher may then properly give attention to the difficulties of individuals. In doing this, the teacher should pass quietly about the room again, stopping this time to give individual attention to those students who have not been able to get started with their work or who are having serious difficulties. This amounts to individual instruction. As in the general developmental work, it should aim to enlist the student's fullest participation in the intellectual task—to guide his thinking rather than to give him ready-made procedures. He should be given only such direct information as may be indispensable in helping him to clear up the concepts, principles, or relationships involved, or in giving him some cue which will enable him to proceed under his own power. In most cases the heuristic method should be used. It takes more time than it does to give information directly, but it is a method which fosters self-reliance and independent thinking, whereas the habitual giving of direct information may actually inhibit the development of these desirable characteristics.

7. Spend no more time than is necessary with a student. On the other hand, do not allow yourself to be rushed through a conference with one student by the importunities of others.

8. Establish with your students the understanding that in general the teacher should determine who needs help, and when, and that, instead of asking for assistance and then idly waiting for it to come, they should persistently continue their efforts to help themselves until such time as the teacher can determine whether they need help or not.

9. Train yourself to detect quickly the key logs in students' mathematical log jams. Not only will this save much time both for the teacher and the students, but it will do much to increase both the interest and the independence with which the students will work. Often the difficulty can be traced to a careless reading of problems or of instructions. At other times mistakes in computation or in fundamental algebraic mechanics may be responsible for the student's confusion. Such difficulties are relatively easy to correct. Often it is not necessary to do more than point out to the student the nature of his difficulty. These cases are very different from those which involve a lack of understanding, and the teacher should be able to determine readily into which class a given case falls. In cases of the latter type it will generally be necessary to do a careful job of reteaching with the individual student.

10. At all times, and above all other considerations, an atmosphere conducive to study should be maintained in the room. Comparative quiet should prevail. While students should not be prohibited from

conferring with one another at times about their work, this should not represent the pattern for the conduct of the directed study period. Each student should be impressed with his personal responsibility in this direction and with the principle that he can best discharge this responsibility by keeping steadfastly at his own work.

Directing study along the lines which have been suggested here is a most illuminating and valuable experience for the teacher. It not only keeps him in touch with the work of all the students, but it often reveals unsuspected omissions or inadequacies in his teaching and is one of the most effective means of keeping his viewpoint adjusted to that of the less mature students. If he is observant, he will learn much concerning their study habits. This should enable him in many cases to help them eliminate unprofitable methods and to substitute for them more effective procedures.

Some Suggestions to the Students on Studying Mathematics. There is abundant evidence that students often employ wasteful and inefficient procedures in studying mathematics. Sometimes they do not know how to begin their work; they waste time in trying aimlessly one thing after another. They fail to form the habit of depending upon themselves. They are unsystematic and do not take time for deliberate reflection before starting their work. They allow their attention to be distracted and their work interrupted. They are careless in their reading, in their listening, and in their written work. Many times they do not recognize that the specific procedures involved in the study of some parts of mathematics are not necessarily the same as those involved in the study of other parts of the subject, and they do not analyze their assignments to determine what particular procedures will contribute most effectively to the mastery of the work in hand. If the students are to acquire the ability to do effective independent study, they need specific instruction in methods and habits of study, and it seems clear that one of the most important tasks of the mathematics teacher is to help them to a knowledge and acquisition of general habits which are conducive to the improvement of study in general, and of specific procedures which are involved in the studying of particular parts or aspects of the work.

The following list of suggestions will be helpful to students and teachers in this connection:

1. Form the habit of studying your mathematics at a regular time.
2. Form the habit of studying your mathematics in a regular place.
3. Form the habit of getting down to work *at once*. Do not dally.

4. Form the habit of paying *concentrated and sustained attention* to your work after you start.

5. Do not allow avoidable interruptions after you start.

6. Work as rapidly as you can after you start.

7. Work by yourself for the most part.

8. As a preparatory step, get the assignment clearly in mind. Recall the teacher's explanation, and, if necessary, study again the sample exercises and the explanations in your text.

9. Plan your work for the work period before you start.

10. Read the problems and exercises carefully. In each case be sure you understand clearly what is given and exactly what you are expected to do, find, or prove. Keep these things clearly in mind while you are working.

11. If you are to copy an exercise or a problem, be sure that you copy it correctly.

12. Take plenty of time to think. Do not start to solve a problem or to make a proof until you have clearly in mind exactly what is given and exactly what is required, and have developed a plan for doing what is wanted.

13. If you do not know how to begin, consult your textbook and try to recall the explanations which your teacher has given.

14. Try to write out the questions that bother you, making them very clear and specific. Often the answer will suggest itself.

15. Do not give up. At least try to find out just where and what your difficulty is.

16. Form the habit of listing all new words and concepts and of learning them at once. Use the dictionary and your text.

17. Form the habit of using the index and the reference tables in your books as sources of information about new words, formulas, numerical values, etc.

18. Memorize important rules, formulas, and facts, but be sure you understand their meanings and can use them correctly.

19. It is easier and better to memorize statements and formulas as wholes than to memorize them by parts.

20. In memorizing formulas it helps to read them aloud.

21. Mnemonic devices are often helpful in memorizing.

22. Work carefully. It is easier to avoid mistakes than it is to find and correct them after they are made.

23. Write neatly. Put down figures in neat rows and columns. Small, round, vertical writing is most legible.

24. Never use scrappy, dog-eared paper. Use scratch paper as little as possible.

25. Remember that every symbol has a definite *meaning*. Always read meanings into the symbols you use.

26. Read thoughtfully, and reflect as you read. Superficial reading in mathematics is generally just a waste of time.

27. Form the habit of expressing verbal statements in symbols.

28. An exercise is frequently made up of a series of steps. Do one step at a time.

29. Compare exercises in algebra with similar types of exercises in arithmetic. Sometimes this will give a cue or suggestion that will be helpful.

30. When time permits, check your work and your answers.

31. In order to fix rules and formulas in your mind, use them as soon as possible after you have learned them.

32. Sketch diagrams or graphs when you can. This often makes it easier to understand problems.

33. In numerical problems form the habit of making a preliminary estimate to serve as a rough check on your work.

34. In preparing for a recitation spend some time in organizing the lesson in a logical form in your mind.

35. Replace large numbers by approximations in planning the solutions of problems.

36. When listening to discussion in class, listen with your whole attention. Do not have books opened or pencils in hand unless specifically asked to do so.

37. Be mentally alert, active, and aggressive.

38. Enjoy overcoming hard mathematical obstacles.

39. In studying material to be understood and digested (as a proof in geometry), plan out the proof or the attack first; then go over it rapidly and sketchily to get it all in mind; next, take it section by section, writing out the details; and finally review it and check it rapidly.

40. Be critical of all statements made, whether by yourself or by someone else. Be especially critical of statements that are not adequately supported by reasons.

By careful training in the systematic use of such a list of study suggestions the students may acquire both the habit and technique of analyzing and improving *their own* study habits. This is the goal toward which we should strive. Too much, of course, must not be expected from this or any other list of study suggestions.[1] It would be utterly misleading to suggest that any such list could be regarded as

[1] The following references contain valuable suggestions for studying mathematics:

Ibid., pp. 87–115.

E. R. Breslich, "Administration of Mathematics in Secondary Schools" (Chicago: The University of Chicago Press, 1933), pp. 349–357.

W. C. Arnold, How to Study Mathematics, *American Mathematical Monthly*, **47** (1940), 704–707.

Ethel M. Hendrick, How to Study Geometry, *School Science and Mathematics*, **30** (1930), 1068–1072.

Esther E. Reese, How to Study Algebra, *ibid.*, **32** (1932), 171–179.

Margaret R. Walters, How to Study Arithmetic, *ibid.*, **34**, (1934), 848–852.

a panacea or as a guarantee of efficiency. On the other hand, it may confidently be expected that deliberate, organized, and systematic attention to the improvement of study habits will in fact return very substantial and gratifying dividends.

TEACHING FOR PERMANENCE: DRILL, REVIEW, AND MAINTENANCE

The developmental and assimilative phases of instruction represent essentially the stages during which actual learning of new material takes place. Any subject matter, however, is likely to be forgotten, no matter how well it has been initially mastered, unless it is maintained by repeated application and practice. This is particularly true of mathematical skills and relationships. Skills need to be perfected and maintained through systematic drill, and concepts and relationships must be reviewed and applied at frequently recurring intervals. The instructional effort which is directed toward these ends may well be called *teaching for permanence.* While it generally involves material that has already been learned rather than new material, its importance as a means of strengthening and maintaining learnings is commensurate with the importance of the developmental and assimilative phases of instruction as means of *acquiring* new learnings. Its avenues are drill, review, and application.

The Function of Drill. The place of drill in mathematics has been a much-debated issue in recent years. The reaction against the excessive and indiscriminate use of drill, which came along with the reorganization movement and with the increased emphasis upon concepts and meanings, has caused some educators to go to the other extreme and to inveigh against all drill as being futile and valueless. The reason for the widely divergent views with respect to the part which drill should play in mathematical instruction lies in the lack of common understanding with regard to the nature of the outcomes of drill and its function in relation to effective learning. The old pedagogy undoubtedly laid too much emphasis upon memorization and mechanical learning, to the consequent neglect of meanings. In such a scheme drill naturally played an extremely prominent part, because it afforded a convenient and efficient medium for the rapid memorization of details and for the automatization of processes. The fallacy in this point of view is the tacit assumption that memorization and automatization imply understanding. This, of course, is not the case. On the other hand, the "new pedagogy," in its extreme form, takes the position that meanings alone have value, and that whatever fails to contribute *directly* to the development of concepts and understandings

has no legitimate place in the educative process. Obviously there would be little place for drill in a program of instruction based upon such a philosophy as this. This point of view overlooks the important element of fixation, without which it would be manifestly impossible to organize and relate concepts or to carry on any process at a reasonable level of efficiency.

An enlightened present-day view of mathematical instruction must reject both of these extreme positions as untenable. Drill must be recognized as an essential means of attaining some of the desired outcomes, just as a strong emphasis upon concepts must be regarded as essential. Many of the operations of mathematics need to be performed not only correctly but with reasonable facility and speed if they are to be very useful. Some of them need to be actually automatized. The acquisition of facility in such operations can be secured only through systematic and repeated practice, *i.e.*, through drill.

If instruction is to be valuable, however, understanding must go hand in hand with operational facility. With a few possible exceptions children should not be drilled on procedures which they do not first understand. Drill under such circumstances lacks both significance and motive. It may indeed produce facility, but facility will be without value unless it is associated with meaning. If understanding and motive are lacking, drill becomes little more than drudgery.

Principles of Drill. Educational psychology in recent years has done much to provide us with well-established principles whereby drill may be made interesting and effective, and authors and publishers have combined to make available materials specially designed to facilitate the application of these principles. In the following paragraphs some of the most important considerations relating to drill in mathematics will be enumerated and briefly discussed.

Drill, to be most effective, must be well motivated. The attitude with which the students approach the problem of mastering material has an important bearing both upon the rate of mastery and the extent of mastery which they achieve with respect to that material. If it is material which contains no intrinsic interest and for which they can see no value, their work will be without interest. However, if they are working on something which they recognize as important or interesting in itself, they will work with enthusiasm and with concentrated attention, and their work will be correspondingly more effective. Contests between selected teams, improvement charts, and games involving the materials to be mastered are typical of the numerous devices which have been developed for motivating drill work in mathematics.

Drill exercises should be conducted in such a manner that students can work at differing rates and at different levels according to their abilities. The certainty of individual differences within a group makes it clear that the individuals, even though they may all need drill on the same things, will not perform at the same rate or at the same level of difficulty. It is uneconomical to have those who have attained substantial mastery continue drilling on tasks which no longer challenge them. It is equally wasteful to have them do nothing while waiting for the others to "catch up." In order to avoid situations of this sort, drill exercises should contain enough material to keep all the students profitably occupied throughout the drill period and also sufficiently diversified material to provide worth-while and stimulating practice for students of different attainments and capacities.

Drill periods should generally be rather short. The attention span of children is not great, and long periods of continuous drill become tiresome and ineffective. In general it may be said that no drill period should extend for more than 20 minutes and that in most cases drill periods of not more than half this length are preferable. This does not mean that a given skill can be mastered to a desired point of proficiency in 5, or 10, or even 20 minutes. It means rather that, if more time than this is required, it should be distributed in relatively small amounts at recurring intervals which should become more widely spaced as time goes on. This principle of spaced learning, as contrasted with the idea of complete immediate mastery, is exceedingly important and is coming to be widely recognized in the organization of textbooks and instructional materials.

In order to be most effective, drill must be specific. By this is meant that it should be concentrated upon particular skills or even on particular details of operation. The students should, of course, be aware of the relation of any detail to the whole situation of which it is a part. But for purposes of fixation, which is the object of all drill, the particular detail or skill should be for the moment dissociated from its setting and context and should be drilled upon per se. When the desired proficiency in the details has been attained, they should be progressively reintegrated into the entire process or situation of which they are components.

When drill is begun on any process or skill, correctness should be insisted upon as the prime consideration, and for the time being speed should be regarded as of secondary importance. Every effort should be made to detect mistakes in students' work and to eliminate them at the outset. Failure to do this will inevitably have unfortunate

consequences, because a wrong habit "fixes" as readily as a right one, and it is much harder to eliminate a wrong habit that has become established and to replace it by a correct one than it is to establish the correct one in the first place. Thus it is of extreme importance to supervise *closely* the initial work of the students on any new process. The insistence upon *right practice* from the start cannot be too greatly emphasized. Teachers often overlook this important principle when they assign homework involving procedures which have not been previously mastered in class. When this is done, the students are likely to make mistakes that could have been avoided by a small amount of carefully supervised drill.

There are few things which cause children to take a keener interest in their work or to apply themselves with more verve and intensity than the satisfaction of knowing immediately whether their work is right or not. In much of the work of secondary mathematics it is desirable to have students apply mathematical tests or checks to ascertain the correctness of their own work. The checking of work in this manner is a real educational exercise, in many cases fully as valuable as the original work itself. This method of applying mathematical checks can be used in connection with drill work just as it can with problem work. It has, however, the disadvantage of slowing up the drill and of diverting time and attention to procedures other than those for which the drill was originally planned.

Some teachers dislike to provide students with answers to verbal problems or materials assigned for home study on the theory that students may easily misuse the answers. It is quite conceivable that this argument may have some justification as regards the kind of work mentioned, but in drill work conducted in the classroom the situation is different. It is, in fact, a definite stimulus to the student to know *immediately* whether his responses are correct or not. If they are, he secures an immediate satisfaction; if they are not, he is challenged to correct his work before his attention has shifted to other things. The question of dishonesty is minimized. The fact that the student realizes that he is being trusted to play fair operates as a definite incentive to him to do just that. Experience has shown that it actually works out in this way and that it affords a real training in honesty and self-responsibility, besides adding zest to the work itself.

Wherever possible, drill materials should also be provided with some means whereby the student can *score* his own work and can compare his performance not only with that of the other members of the class but also with established standards and with his own performance on

previous occasions. Here again, experience has shown that students are greatly interested in noting their own progress, and no finer incentive than this could be devised. The most valuable of the published drill materials are those for which standards of comparison are available and which are provided with record sheets or charts on which each student can keep a continuous running record of his own achievement.

Finally, drill must be well oriented in, and related to, an instructional program designed to emphasize understandings, appreciations, and generalizations. Too frequently the tendency to overemphasize proficiency in mechanical skills is the outstanding characteristic of a mathematical drill program. Speed and accuracy in the fundamental skills are very desirable goals of mathematical instruction, but the truly functional program will at all times also emphasize the careful study of interrelationships, the intelligent comprehension of underlying truths, and the thoughtful generalization of principles and processes. Definite opportunity should be offered for reflective practice on these more intangible aspects of mathematical learning, as well as on the more tangible concepts and mechanical skills.

Review. Review is sometimes mistakenly identified with drill because they are both characterized by repetition and because they both aim at the fixation of reactions, concepts, or relationships. In spite of these common characteristics, however, it is a mistake to regard their functions as identical. Drill is concerned chiefly with the automatization of relatively detailed processes and reactions. Review, on the other hand, has a dual function. It aims not only at the fixation and retention of facts, processes, and concepts, but also at the thoughtful organization of the details of subject matter into a coherent whole in order that the relationship of the various parts to each other and to the whole may be clearly understood. Review is usually concerned with more or less comprehensive units of subject matter, whereas drill is generally upon details. One of the functions of review is to make recall more certain and more effective, but the fulfillment of this function demands more than mere remembering. Review aims at fixation and retention, but it aims to achieve these through the deliberate processes of organizing, systematizing, and relating elements and of generalizing and applying principles rather than through reducing reactions to the plane of automatic responses. Thus, while drill and review have certain things in common, they also have certain important differences. Each has its proper function, and each is exceedingly important in the study of mathematics.

Review work may be incidental in the sense that it may be integrated

with the other work of the course, or it may be specialized by making it the primary feature and objective of particular assignments. Both of these types of review are necessary to the most effective teaching. Review of the incidental type is especially valuable for the gradual building up and clarification of concepts through repeated reference and through continual reapplication in those situations in which they play component parts. Concepts and principles are generalized through being met with in many situations which vary in other particulars and from which the concepts and principles are gradually dissociated and abstracted. Perhaps this process may not be recognized as review at all if it is systematically made an integral part of the regular work, but it is review in a very real and important sense. One of the strong arguments for a continuous program of integrated mathematics is that this sort of incidental or integrated review would necessarily run systematically throughout the entire program, giving strength and coherence to the entire structure through continual interassociation of the components.

At the same time there is need of special review work to supplement the incidental review which has been described. The functions of the special review are to help the student organize more or less comprehensive bodies of material with reference to their logical relationships, to assist him in classifying their important ideas, and to give him a sense of the unity of the whole which might otherwise be lacking. The "review lesson," which should be planned with this idea dominant, will generally follow the assimilative study of a unit. In preparing for such a review lesson, the student should be expected to summarize the outstanding ideas which have been considered in the unit and to make an outline from which he can give a brief but coherent and systematic discussion of the material in the unit or division. The preparation of such an outline will make it necessary for the student to review the unit in the fullest sense of the term. Through making the necessary association of the ideas in the unit, he will be aided not only in remembering them but in understanding them and appreciating their interrelations.

On the whole, most teachers do a better job of conducting drill work than they do of conducting review work of this type. This is probably due in part to their failure to recognize the main function of review as different from that of drill. Students need to be taught how to review material just as they need to be taught how to study. They cannot review effectively without definite instructions. Yet all too commonly the only instructions they receive are "Review chapters seven and

eight for tomorrow." The task of helping students plan their review work is a responsibility which every teacher should take seriously.

Maintenance. A planned program of cumulative drill and review work is aptly designated as a maintenance program. The importance of such a program is implied in the discussion in the foregoing paragraphs. The fundamental requirement of a satisfactory maintenance program is that it shall operate to prevent the forgetting of facts, concepts, and relationships and to forestall the disintegration of skills. To this end it must provide for systematic application of the important elements of the instructional program, and for appropriate or needed practice on these elements even after current attention and emphasis have passed on to other matters. Therefore the planning of a really adequate maintenance program must be built upon the following principles:

1. The materials to be included should be selected from the point of view of relative values. The program should not be cluttered up with trivial things. Only significant skills, concepts, relationships, principles, and problem situations should be included.
2. In accordance with established principles of drill and review, the items should be distributed throughout the program in such a way that practice upon any particular element will not be too greatly concentrated but will recur at increasing intervals and in decreasing amounts.
3. The maintenance program should be diagnostic, preferably self-diagnostic, so that each student may be able to discover his own weaknesses. To this end some means should be provided whereby each student may systematically keep and study his own achievements in detail.
4. There should be provided supplementary practice material for remedial work on the various particular elements included in the maintenance program. This supplementary material can be used most effectively if it is keyed with the diagnostic record. In this way each student will be able not only to determine those things upon which he most needs practice, but to carry on his own remedial work with a minimum of direction.
5. The different sets of exercises in the maintenance program should be comparable in terms of some uniform scoring or rating schedule, so that each student may keep a record of his general achievement and his progress. This will be of great value in stimulating pride, effort, and genuine interest in maintaining skills and principles after the original interest due to their newness has worn off.

Numerous textbooks published in recent years recognize the need for systematic maintenance work and make provision for it through sets of drill exercises, diagnostic inventory tests, cumulative reviews, and the like, placed at strategic points in the texts. In some cases these

exercises have evidently been prepared hastily and with little attention to their validity or suitability. In other cases their organization and arrangement have been based upon extensive and painstaking study and upon well-established principles of the psychology of learning. These same comments are equally applicable to the multitude of drill books and workbooks and sets of practice exercises which are now commercially available to supplement textbooks. They are not all equally good, but the better ones are valuable aids to the teacher in carrying on an adequate maintenance program. Many teachers lack both the experience and the time needed to prepare thoroughly suitable materials for regular maintenance work. Prepared materials which are scientifically planned and for which standards of attainment are available serve at least three useful purposes: (1) they make for economy of time and labor and therefore for efficiency in instruction; (2) they provide a strong motive to achievement, since they foster the students' continued study of their own performances, and (3) they provide the best possible insurance against forgetting and against the deterioration of skills and understandings.

TEACHING FOR TRANSFER

The status of transfer values has been discussed in a previous chapter where it was pointed out that the likelihood of transfer resides not so much in the subject matter as in the ideals and attitudes inspired by the teacher and in the methods used in teaching. In other words, the transfer of training is a legitimate objective for which to work in the teaching of mathematics, but its achievement is not be be looked for with any degree of assurance unless the teaching is definitely planned and carried on with this particular end in view. The problem of teaching for transfer is therefore dependent upon the following questions:

1. What are those elements of mathematical training the transfer of which to other situations is desirable?
2. By what methods of teaching can the transfer of these elements be fostered and promoted most effectively?

The Objects of Transfer. The first of these questions can be answered with definiteness. In the first place, it is desirable that all those elements of mathematical training which most people have occasion to *use* shall be taught in such a manner that they *can* be used whenever occasion demands. This category includes such things as the fundamental combinations, skills, operations, and concepts of

arithmetic; the laws and formulas involved in mensuration of the common geometrical figures; the interpretation of commonly used statistical conventions and devices; the construction and interpretation of straight-line, circle, and bar graphs; the ability to read pictographs intelligently; the fundamental meaning of a formula; the ability to evaluate formulas; in a word, practically all the understandings and abilities, other than those of formal algebra, which are commonly included in the mathematics of the junior high school. These things are fairly specific and are needed by practically everybody. Since they cannot be directly taught in all the specific situations to which they have potential application, it is desirable that the generality of their application be emphasized so that the student will not be at a loss when occasion requires their use in new situations.

In the second place, the fundamental concepts, formulas, and skills of elementary algebra are desirable objects of transfer Too often these are taught with specific reference only to the immediate algebraic situations in which they occur in the textbook and with little or no reference to the generality of their meanings or applications. To this group of algebraic understandings and skills should be added the knowledge and understanding of certain of the more important facts and relationships that are developed in plane and solid geometry. Of course, every proposition constitutes a link in the immediate chain of development, and to this extent the very consciousness of its relation to the preceding and subsequent parts of the development involves a measure of transfer, but some of the propositions constitute extremely important generalizations which have wide application not only in the field of demonstrative geometry itself but also in subsequent mathematical courses and in other fields of study such as engineering and certain parts of the physical sciences. Substantially the same observation may be made with reference to the concepts, skills, and relationships of numerical trigonometry. Such important and pervasive generalizations as the Pythagorean theorem, the angle-sum relationship, the proportionality of line lengths in similar figures, the sine and cosine laws, the metric properties of circles, and various area formulas are cases in point.

The foregoing list of objects of transfer may be grouped for convenience into two categories, *viz*, things to know and understand and things to be able to do. The detailed items which would be included under either of these lists are reasonably specific.

A second type of objects of transfer is represented by broader and more abstract concepts. These are well illustrated by the headings of

Chaps. IV through X of the 1940 Report of the Committee on Mathematics of the Progressive Education Association.[1]

From a reading of these seven chapters it becomes evident that the committee which prepared the Report had in mind one general aim and that was to emphasize the proposition that the study of mathematics should aim to bring about a consciousness of the general nature of these concepts and of their general applicability to specific situations. In other words, the aim, expressed or implied, was to emphasize the importance of developing these broad and abstract concepts with a particular view to realizing their transfer possibilities.

Finally, there is a third object of transfer which receives much emphasis in writings about transfer of training but little in actual teaching. This is the acquisition of a mathematical manner of thinking. It is implied in the statement of the "disciplinary aims" listed in the Report of the National Committee on Mathematical Requirements, "The acquisition of mental habits and attitudes . . . ,"[2] and it is precisely the thing which all mathematics teachers hope for but which, for lack of a methodology, has been anticipated, if at all, as a by-product of mathematical instruction rather than as a general but definite outcome to be worked for through definite procedures. It is encouraging to note that at least in the field of geometry beginnings have been made in the evolution of a methodology which appears to hold considerable promise of success in this direction.[3]

How Shall We Teach to Secure Transfer? Students of modern educational psychology are agreed that transfer is not automatic. Furthermore, there seems to be a trend away from the rather simplified concept of transfer that is implied in Thorndike's theory of identical elements. The theory of generalization advanced by Judd probably represents more correctly the avenue through which positive transfer of higher mental functions takes place. This theory does not deny the importance of identical or similar elements. In fact it implies their necessity but denies their sufficiency to account for the phenomenon. It says, in effect: If a principle is to transfer to (or be applied intelligently in) a particular situation, the situation must, of course, contain elements or relationships analogous to those found in the principle, but

[1] See p. 40.

[2] Report of the National Committee on Mathematical Requirements, "The Reorganization of Mathematics in Secondary Education" (Boston: Houghton Mifflin Company, 1923), p. 12.

[3] For a description of experimental work along this line see the *Thirteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1938).

this is not enough. These similarities not only must exist; *they must also be recognized by the learner before significant positive transfer can take place.* It is this act of recognition of similar elements which really constitutes transfer at the higher levels and which, indeed, alone characterizes all functional and relational thinking and sets it apart from mere specific identification and mechanical rule-of-thumb procedure.

The problem of teaching for transfer would seem to resolve itself, then, into the problem of teaching children, not only to recognize similarities between new situations and other situations with which they are already familiar, but to form the habit of consciously being on the lookout for these similarities. When confronted by a new and unfamiliar situation, the student must learn to ask himself, in effect, "Does this situation fit into the pattern of any other experiences or situations with which I am already familiar? What elements of similarity are there, and how can I use these elements that are familiar to me in interpreting this new situation and in bringing it under my control?"

A typical illustration of lack of transfer is found in the inability of many students to apply the principles and operations of algebra to problems in physics. The physics teachers usually complain that the students have not mastered the mathematical principles involved, but in most cases it is more likely that they simply fail to recognize in the concrete physical problem relationships which are perhaps quite familiar to them when seen in the abstract or symbolic mathematical setting in which they have been encountered previously. Innumerable examples illustrating this point could be given. They occur many times even within a particular branch of mathematics itself. Thus students who will readily factor $a^2 - b^2$ may fail to recognize such expressions as $t^2 - 36$ or $x^2 - 2xy + y^2 - 9z^4$ as being of precisely the same type and so may be unable to factor the latter expressions. Similarly, in connection with verbal problems, which bear the reputation of being the hardest part of elementary algebra, the difficulty is almost never in solving the equations to which these problems give rise but rather in translating the verbal problems into symbolic form. This is mainly due to failure to recognize the essential identity of the abstract and generalized symbols, formulas, and equations of formalized algebra with the concrete and specific conditions and relationships set forth in the verbal statements of the problems.

In some cases, of course, the similarities or identities in different situations are simple and obvious, and in such cases transfer is fairly

well assured, especially among students of superior intelligence. In many cases, however, the similarities are obscured by other more prominent elements and, in such cases, it is often necessary to make careful analyses in order to disclose them. Children will not learn to make these analyses unless they are systematically trained to do so. They need to be shown how to make them and to have much practice under carefully supervised conditions in order to master the technique. But even this is not enough. If this practice is to become really functional in their mathematical training, they should become impressed with the advantage of *habitually* making this approach to any problem. The student should form the habit of deliberately instituting a search for elements or relationships in the problem in hand which are similar to corresponding elements or relationships in other situations with which he has already had experience, whenever such similarities are not apparent at the outset. To the extent that such a procedure is consistently followed, the transfer of mathematical processes and techniques will be facilitated and this essentially mathematical mode of thought will become a really functional contributor to the effectiveness of rational thinking in general.

PROVISION FOR INDIVIDUAL DIFFERENCES

It is axiomatic to say that, if instruction is to be really effective, it must reach the individual students, and individuals differ greatly in their interests and in their abilities. The problem of adapting instruction to individual differences has existed whenever and wherever the group receiving instruction has consisted of more than one student, but since early in this century it has occupied a much more prominent place in the attention of the educational world than it had ever occupied before. There are several reasons for this; the most important one is that the problem itself has become much more acute and pressing in the secondary schools than it was before. The unprecedented and bewildering growth of the secondary school has been accompanied by a decline in the average intellectual ability of the student population; the spread of the range of abilities has been correspondingly accentuated. As a result, the always questionable practice of giving identical instruction to all students in an unselected group has become more questionable than ever. If the instruction for the group is geared to a level which will challenge the abilities of the better students, then those of mediocre ability will miss much of it and will tend to lose interest or will resort to memorizing, while the inferior students will soon fall hopelessly behind and become discouraged. On the other hand, if the

instruction is adapted to the limited abilities of the slow students, then the superior students will soon lose interest because the work will not challenge their best efforts. In either case the situation not only will result in inefficient instruction but may easily become a fertile breeding ground for serious disciplinary problems.

The educational world is now keenly aware of this problem and of its implications. It has recognized that the only effective method of meeting this educational dilemma is through differentiation of instruction and requirements in accordance with the capacities of students. Recent courses of study, professional books and articles, and the prefaces of nearly all recent textbooks in secondary mathematics bear witness to the urgency of the problem and to the intense effort which is being put forth to provide suitably differentiated materials and methods of instruction, to the end that profitable work may be provided for all students.

This attempt to provide for individual differences has taken a number of forms, prominent among which may be mentioned ability grouping or homogeneous grouping, differentiated assignments, honors courses, directed study, and even individual instruction. The latter plan, although theoretically desirable, if viewed solely from the instructional standpoint, largely precludes the valuable social element which can be had only in group instruction. It is subject to such obvious practical limitations that it can never become widely used as a normal procedure in our secondary schools. The other plans, however, are all more or less adaptable to school conditions as they exist.

Ability Grouping. The plan which most people have come to associate most readily with provision for individual differences is the arrangement generally called "homogeneous grouping" or "ability grouping." As the name implies, it consists essentially of grouping the students in such a way that disparities in the abilities within a given group will be reduced as far as possible. The great objection to the traditional miscellaneous grouping has been that instruction inevitably becomes geared to some one level of ability to the consequent disadvantage of all students whose capacities are either above or below that particular level. The plan of ability grouping, while it would not entirely eliminate individual differences within any group, would materially reduce the range of abilities within each group and thus tend to minimize the seriousness of the problem.

There have been numerous adaptations of the plan, but its basic principles and *modus operandi* are well defined. Educators are not entirely agreed as to its effectiveness or its desirability; in fact, it has

sometimes been charged that the plan is undemocratic and that its psychological effect upon the students is not wholesome. In particular, some individuals feel that it accentuates a feeling of inferiority on the part of the weaker students. Others, however, deny this and assert that the plan is more democratic than unselected grouping in that it facilitates the adjustment of each student's work so that incentive will be increased and students not only enabled to work at their own optimum rates but encouraged to put forth their best efforts. The fact that this plan has been used for a number of years in many of the larger cities would seem to indicate that it is meeting with some measure of success and that its advantages probably outweigh any disadvantages which may attend it.

In the past the differentiation of students has been based mainly upon intelligence quotients. There is a feeling in many quarters that this is not an entirely satisfactory criterion and that intelligence ratings should be supplemented by prognostic tests in mathematics, by marks in former courses, and by teachers' estimates of probable success in subsequent mathematics courses. The most serious administrative limitation of the plan of ability grouping is that it is of no use to small schools since it can be used only in schools where there are enough students to justify two or more class sections in the same subject.

Differentiated Assignments. A method of adapting instruction to individual differences and needs *within* a class group is the use of differentiated assignments for students whose abilities or rates of work are not alike. This plan has met with favor since it can be used in schools which are too small to permit homogeneous grouping. While it tends to complicate the work of the teacher, it has much to commend it from the standpoint of instructional effectiveness. The "contract" type of assignment is a good illustration of this plan. Under this type of assignment each unit of work is organized in such a way that the accomplishment required for a bare passing grade is specified as the minimum "contract" which all students are required to execute. Other "contracts" containing additional work of a more difficult nature are set up as requirements for successively higher marks, each contract being gauged to a higher level of accomplishment than the preceding one. The illustrative example shown on page 189 was used as the contract for a particular course of study.

The contract plan is exceedingly definite so that the student may know at any time approximately where he stands. It has two disadvantages, however. One of these is that superior students are generally required to execute all the details of the minimum contract before

Suggested Form of Contract for Seventh Grade Mathematics; Second Month; Second Semester

D Contract

1. Be able to find the area of any square, rectangle, parallelogram or triangle, when dimensions are given
2. Hand in at least half as many problems as the average of the class.
3. Take all required tests.
4. Have an average rating of at least 5 on your workbook.
5. Be able to find the volume of any cube or rectangular prism, when the dimensions are given.
6. Be able to estimate the area of an irregular-shaped figure by using graph paper.
7. Have a good attitude and be pleasant and industrious.
8. Meet any other requirements that may be added and posted here

C Contract

All the *D* Contract plus the following:

9. Be able to find the area of any trapezoid, dimensions being given.
10. Be able to find the perimeter and area of any circle if the radius or diameter are known
11. Solve problems based on areas and volumes of figures mentioned above.
12. Hand in at least four-fifths as many problems as the average of the class.
13. Have an average rating of at least 6 on your workbook.

B Contract

All the *C* Contract plus the following:

14. Be able to *explain how we got the formulas* for the areas of the triangle, parallelogram, and trapezoid.
15. Hand in *more* problems than the average of the class
16. Have an average rating of at least 7 on your workbook.
17. Hand in at least one acceptable out-of-class project.

A Contract

All the *B* Contract plus the following:

18. Hand in 20 per cent more problems than the average of the class.
19. Have an average rating of at least 8 on your workbook.
20. Make an outline of notes covering the important things in the month's work and be prepared to give a 5-minute review discussion of this, using only the notes as your guide.
21. Take a private 10-minute oral examination.
22. Hand in at least three acceptable out-of-class projects.

passing on to the higher ones, and often much of the work of the minimum contract is rather simple and monotonous routine which fails to interest or challenge these more capable students. The other disadvantage is that the preparation of the various contracts in suitable form for the students to use and the large amount of record keeping

which is necessary place a severe burden of extra work on the teacher and may thus impair his effectiveness in the actual instructional work. This is especially likely to happen in cases where the facilities for mimeographing or duplicating assignment sheets and other work materials are inadequate.

The contract plan, however, is but one method of providing for individual differences within a class. Another method which is somewhat less definite but which is probably used more widely is the assignment of special projects or reports or of particularly difficult problems to students who have given evidence of superior ability. This method emphasizes quality and caliber of achievement rather than mere quantity. In most cases it is easier to administer than the contract plan, and it avoids some of the objections of the latter plan. There is much to be said in favor of this method of differentiated assignments, and a majority of the recent textbooks in secondary mathematics recognize its potential value by including numerous topics, problems, and exercises designated as being optional but suitable for students of more than average ability. Some textbooks go so far as to classify all the exercises and problems into three categories of difficulty to correspond to the three-way classification customarily employed in homogeneous grouping.

Somewhat allied to both of the plans which have been mentioned, but at the same time differing from both of them in certain respects, is the proposal for so-called "honors courses." The underlying idea of the honors course is to relieve particularly brilliant students of unnecessary tedium and waste of time and at the same time to challenge their best mathematical efforts. It recognizes the fact that there are occasional students for whom much of the normal work of the class is easy to the point of being boresome, and it proposes to afford the opportunity for such students to direct their efforts toward special problems which lie definitely beyond the normal scope of the course. In other words, it proposes to offer the brilliant student the opportunity for original and largely independent study of special mathematical topics not contemplated for general class study. Honors courses of this nature have not come into very general use probably because they are off the beaten track and involve extra work in individual supervision and planning on the part of the teacher. As a means for capitalizing the abilities of the very superior students, however, the plan has much to commend it. In cases where it has been given a fair trial, it has more than justified itself in the mathematical growth and the stimulation of interest which have resulted.

Directed Study and Individual Differences. Directed study offers a third means of providing for individual differences among students. This is one of its functions, and it operates toward this end in two ways. In the first place it provides a means through which all students may work at their own individual optimum rates and whereby conditions are provided that are favorable to the exercise and development of initiative and of individual abilities for independent work. Secondly, it provides conditions under which those students who find themselves in need of help may secure such help at the time when they need it and directly from the teacher.

It would be a mistake, however, to infer that directed study as it is commonly carried on is equally advantageous to all students. There is now a substantial body of experimental evidence which indicates that the less capable students are generally benefited under a program of directed study but that the brighter students tend to do less well than they do under the traditional plan. While this may at first seem surprising, reflection will convince one that it might reasonably have been expected. The whole movement for taking care of individual differences, of which directed study is a part, has been from the beginning primarily concerned with the welfare of the weaker students, and in the efforts to develop means of providing more adequately for the needs of this class of students the needs of the superior students have received comparatively little consideration

The technique of directed study as it is generally conducted is a technique which stresses assistance by the teacher. Such assistance is proper when it is really needed, but it is improper and destructive of self-reliance if it is given when it is not really needed. It is not always easy, however, for the teacher to determine the merits of a case, and in cases of uncertainty most teachers tend to err on the side of giving help rather than of withholding it. Where help is available for the asking, the capable student may be tempted to take the easiest way and avail himself of it even though he may be quite capable of getting along on his own resources. Thus, although with the best of intentions, the teacher may unwittingly do the capable student a serious injustice by allowing his self-reliance to be undermined. The problem of keeping the work of the superior student at a level which will really challenge him is important in directed study, and, while the needs of the weaker students may be more obvious, this problem should be kept in mind continually in order that the superior students, as well as those who are less capable, may profit under the plan.

However, the fact that in the past the plan has worked to the dis-

advantage of the superior students must not be taken as an indictment of the underlying philosophy of directed study. Rather, it may be attributed to the fact that the obvious and pressing needs of one type of student have operated to draw attention away from the less obvious, but not less real, needs of students of a different caliber. If the plan is to work successfully for all, it will be necessary to administer it through differential techniques adapted to the needs of students of different degrees of ability.[1]

The Role of Prognosis and Diagnosis in Providing for Individual Differences. There is another important aspect of the general problem of providing for individual differences which deserves special consideration. This is the predetermination, or rather the pre-estimation, of the probable success of students in their mathematical work and the consequent guidance of these students in the selection of courses. Of equal importance is the subsequent identification of difficulties and the provision of remedial measures designed to obviate or minimize these difficulties and to set the students on the way to successful accomplishment. The role of prognosis has been implied in part in the discussion of ability grouping, but the whole discussion of providing for individual differences would be incomplete if it did not include specific consideration of the functions of both prognosis and diagnosis with their implied techniques of guidance and remedial work.

There can be no doubt that students often are enrolled in courses in which their expectation of real success and profitable achievement is doomed at the outset by lack of ability. On the other hand, students often are permitted to avoid courses for which they have ample ability and from which they could derive substantial benefit. Either of these situations represents educational waste which could be prevented to a considerable extent by means of wise guidance based on careful prognosis. Of course, it must be admitted that the best prognostic instruments which are available at present are far from perfect, but they are much better than none. If they were systematically used as a basis for a proper guidance program, many of the problem cases calling for special later adjustment might be avoided at the outset. Certainly such a program should be regarded as an important phase of providing for individual differences, because it implies the salvaging of interest, the conservation of personality values, and the prevention of educational waste.

Similarly a systematic program of diagnosis and appropriate remedial

[1] See E R Breslich, "The Technique of Teaching Secondary-school Mathematics" (Chicago: University of Chicago Press, 1930), pp 38*ff*.

work must be regarded as of extreme importance in this connection. Such a program, if systematically carried on, can do a great deal toward the prevention of scholastic delinquency and discouragement and toward the maintenance of interest and the promotion of success. Frequently it is possible to trace maladjustment and failure back to particular causes such as poor reading ability, lack of motive, excessive absorption in other interests, inadequate mastery of technical vocabulary and fundamental concepts, etc. By proper attention to these underlying causes it may be possible to bring hope out of discouragement, order and understanding out of confusion, and success out of failure. Far more could be done along this line than is ordinarily done, and it may be expected that the school of the future will insist that diagnostic and remedial work go hand in hand with prognosis and guidance in the effort to achieve an optimum adjustment between the individual student and his work.

PLANNING FOR EFFECTIVE INSTRUCTION

No part of the work of a mathematics teacher is of more importance than the planning of his work. It is not altogether unnatural that many teachers give far too little attention to careful planning, because the heavy instructional loads which most teachers carry, together with the various extracurricular responsibilities which they are expected to assume, leave them with very little time for reflective consideration of anything but the exigencies of the moment. Nevertheless, it is unfortunate that any circumstances should prevent the careful planning of work. To be able to capitalize situations as they arise in a class is an inestimable asset to any teacher, but it is never wise nor safe to trust entirely to the inspiration of the moment. Careful planning is the only insurance which teachers can provide against waste and inefficiency in their work.

Long-range Planning. There are three main levels or stages of planning. The first of these is the rough layout of the work for an entire year or semester. This involves a survey of all the work to be covered, the organization of this work in terms of units or chapters, the assembling of these into what appears to be the most appropriate sequence, the determination of approximate time allotments, and the allocation of time for reviews and tests. It involves also the formulation of a general testing program for the year or the semester.

A general broad layout of this nature is valuable because it necessitates the consideration of the various units of subject matter from the standpoint of their relative values and because it provides an approxi-

mate schedule for the work, by means of which progress may be regulated and a balanced emphasis maintained. In the absence of such a schedule an inordinate amount of time is sometimes spent on the earlier parts of a course, and where this occurs the later parts of the work are likely to receive hasty and superficial treatment. The plan for the semester or the year should generally include the list of units to be studied, with the time schedule given by weeks. Such an outline provides the teacher with a standard of reference at all times, and, by frequently checking actual progress against this reference schedule, it is possible to maintain a fairly uniform and balanced rate of progress.

The second stage comprises the detailed planning of the separate units of work, each of which may cover one day, several days, or maybe several weeks. The planning of such a unit requires careful and detailed analysis of the material; the formulation of the general and specific objectives for the unit; the selection, rejection, and arrangement of topics and activities; the provision for all necessary tests; and the establishment of a suggestive, though not specifically binding, time schedule.

The actual preparation of the plan for each unit of work will do much to clarify in the teacher's mind the functional goals which he wants the class to attain and to help him to view the entire unit as an *organized* body of subject matter rather than a mere assemblage of more or less unrelated details. It is also important in that it forces the teacher to compare the different topics and details within the unit with reference to their relative importance, thus giving a basis for wise selection and appropriate emphasis of the subject matter to be included within the unit. It also compels the teacher to take into consideration the relative difficulty of the various parts of the subject matter and, in this way, facilitates the preparation of differentiated assignments in adjusting the requirements of the course to different levels of ability among the students. One of the greatest advantages of planning a unit as a whole is that this procedure makes for effective presentation of the unit. With the work definitely planned and organized, the students can be given a coherent preview of the entire unit. This, in turn, makes the developmental work more meaningful than it would otherwise be and adds understanding, interest, and motive to the activities of the students during the subsequent period of assimilative study.

Planning the Daily Lesson. The third and final stage is that of daily lesson planning. This consists mainly of the preparation of an orderly sequence of activities designed to contribute directly to the attainment of specific objectives. It implies attention to such things

as effective and economical classroom management and routine, special drill, review, testing activities, developmental work, making assignments, directing study, and any special activities to be carried on during the class period. It is perhaps even more important than the other two stages of planning which have been discussed because it has to do directly with the immediate activities of the class period, and it is mainly upon the successful prosecution of these activities that ultimate success depends.

Analysis of the particular activities which normally occur in well-conducted mathematics classes would yield substantially the following list:[1]

Routine classroom management activities (adjusting the lighting or ventilation, taking the roll, collecting or distributing papers, etc.)
Preview and developmental work
Reteaching topics inadequately mastered
Assignment of work to be done
Directed study (may include blackboard work)
Drill and review
Special activities (reports, laboratory or field work with special instruments, projects, etc.)
Testing

Before making out a daily lesson plan, the teacher should carefully think through the main things he wants to do and the things he wants the students to do during the class period for which the plan is to be made. These should be set down precisely in the order in which the teacher wants them to occur, with estimated approximate time allowances. The lesson plan should be neither perfunctory nor stereotyped but should be adapted from day to day in such a way as to take full advantage of the educational possibilities of the class situation. Obviously different activities will receive special emphasis on different days. On some days most of the time will need to be spent on developmental work, while on other days the main activity will be directed study, and still other days will be given over largely to testing. The wise variation of the class-period activities is a major factor in stimulating interest and in preventing boredom and disciplinary difficulties. It is possible, however, to use a general outline form which will be objective enough to serve as a useful guide in planning and, at the same time, will be sufficiently flexible to permit adaptation of the lesson plan to any class situation.

[1] No significance whatever is to be attached to the order in which these items are listed here.

A final word of caution should be given with reference to the use of lesson plans. In planning the work of a class period the teacher must of necessity work on the assumption that the activities of the period will follow a definite course without interruption or deflection. All experienced teachers know, however, that this assumption is often wrong. Circumstances which cannot be foreseen inevitably arise at times, and often such circumstances make it advisable for the teacher to depart from his prepared plan. If by doing this it is possible to capitalize some unexpected situation and thereby to stimulate the interest of the students in their work or to repair some unsuspected weakness, the teacher should not hesitate to divert the activities of the period from their charted course. Normally, of course, the best results will be obtained by following the prepared plans, but the teacher should not feel obligated to follow them with a slavish fidelity which would forbid him to take advantage of such opportunities. Frequently, the most effective teaching may be accomplished through spontaneous teacher reaction to unexpected student problems and unpredicted teaching situations.

Exercises

1. Make a clear contrast of the fundamental aims of the three phases in the instructional process: developmental teaching, teaching for assimilation, teaching for permanence.

2. Explain why all three phases are complementary and necessary parts of the whole learning process.

3. Point out the harmful effect that would result from inadequate or ineffective attention to any one of these three phases of the instructional process.

4. Which of these three phases in the instructional process is, in your opinion, most neglected? Give your reasons for your answer.

5. Discuss the function of the "inventory" and the "preview," and explain why both are important in the teaching of mathematics.

6. How and why might the inventory affect the teaching of a unit?

7. How and why might the preview affect the learning of a unit?

8. What is meant by developmental teaching?

9. What suggestions are made in this chapter with reference to the program of developmental teaching in mathematics?

10. Proponents of the laboratory method of teaching in mathematics argue that certain outcomes can best be attained by this method. What are these outcomes, and what are the valid arguments in favor of this method? Be specific.

11. In what grades or in what (mathematical) subjects is the laboratory method likely to be most effective? Why?

12. Discuss in some detail any disadvantages or limitations inherent in the use of the laboratory method, especially in the sequential courses of senior-high-school mathematics.

13. Enumerate the advantages claimed for directed study as an integral part of the teaching program in secondary-school mathematics.

14. What are some of the more important dangers which must be guarded against in any program of directed study?

15. How do you account for the fact that superior students seem to profit less from directed study in mathematics than do slower students?

16. What might be done to make directed study more profitable for superior students?

17. In what specific ways do you think the list of suggestions on studying mathematics, given in this chapter, could be made helpful to students?

18. In what ways could this list be helpful to teachers?

19. Give an illustration of a poor study assignment in algebra or geometry. Why is it poor? Improve it, and show why your improved version will be more helpful to the students in their study of the assignment.

20. What is the valid function of drill, and why is drill necessary? Be specific, and give illustrations to show what you mean.

21. Enumerate some of the most important principles of effective drill procedure, and illustrate each.

22. What is the valid function of review? Why is review important, and for what purposes should it be used?

23. Explain clearly and illustrate what is meant by "incidental review" as it is discussed in this chapter.

24. Show why incidental review needs to be supplemented by special reviews at various times.

25. What is meant by a maintenance program as the term is used in this chapter? Explain what is involved in setting up such a program, for example, in ninth-grade algebra.

26. Compare and rank several textbooks in ninth-grade algebra with respect to the adequacy with which they provide for systematic maintenance work.

27. Give illustrations of how workbooks in arithmetic, algebra, or geometry may be used to provide ready made maintenance programs.

28. Describe how the transfer of a "mathematical manner of thinking" was accomplished in the experiment reported in the *Thirteenth Yearbook* of the National Council of Teachers of Mathematics.

29. Summarize the discussion of the question "How shall we teach for transfer?" Bring out the main points in the discussion.

30. Give illustrations, selected from algebra and geometry, of the fact that transfer takes place whenever a student is able to recognize a particular problem situation as being a special case of a previously generalized relationship and to apply the generalized principle to the particular case in question.

31. Justify or criticize the assertion that the traditional difficulty which students have with verbal problems in algebra is mainly due to lack of transfer.

32. Discuss the advantages and disadvantages of the following methods of providing for individual differences: individual instruction; opportunity rooms; homogeneous grouping; differentiated assignments; directed study; special honors courses.

33. Discuss the role of prognosis and guidance in providing for individual differences.

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CHAPTER VIII

EVALUATION OF INSTRUCTION

The effectiveness of instruction is usually determined by means of checking accomplished results against objectives undertaken. Measures of achievement have thus long been employed as an integral part of the educational program. The present era, which has seen the birth and development of the standardized test and the concomitant testing movement, has also witnessed the evolution of a new philosophy of evaluation. An efficient program of evaluation no longer consists merely in the effort to check the completed process but rather in the continual appraisal of student progress toward the attainment of pre-established aims. Such a program should be outlined in terms of significant instructional objectives and used for more efficient pupil guidance. There is probably no more accurate barometer of the fundamental philosophy of any curriculum than a careful analysis of its evaluation program; the techniques used; the aims, objectives, and functions implied; and the interpretation and use of obtained results.

Nature and Purposes of Evaluation. The emergence of pupil guidance as a significant responsibility of every educational program has placed new emphasis on the function of evaluation. No longer is this function defined in terms of the mere measurement of achievement of ill-defined standards. Standards of achievement have been more clearly defined, more carefully differentiated, and better adapted to different kinds of capacity and individual levels of attainment. Furthermore,

> . . . the scope of testing has been greatly extended and an ever larger group of teachers has become concerned with the evaluation of more than subject-matter achievement. They recognize that mastery of various bodies of subject content is but one aspect of education; and they are attempting to evaluate the development of interests, appreciations, and other characteristics of personality to which the schools are increasingly directing their attention. In this connection it is important to note that evaluation means more than the giving of tests or examinations: the term is used to refer to any method of obtaining and interpreting evidence about the development of pupils.[1]

[1] Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics, The Place of Mathematics

Thus, evaluation becomes an important function in the educative process. It is no longer to be considered merely as a separate procedure to be used at convenient intervals for the purpose of determining marks but as a continual process closely related to each element of the curriculum. The major responsibilities of such an evaluation program may be grouped as follows:

1. To help provide more intelligent guidance of teaching and learning
2. To develop more effective curricula and educative experiences
3. To secure more intelligent and effective cooperation with parents and community
4. To provide an adequate and objective basis for reporting progress[1]

What are the characteristics of a satisfactory program of evaluation?[2] It must be *comprehensive*, *flexible*, and *balanced*. The program should be designed to measure more than the mere recall of information. Upon the teacher of secondary mathematics is placed the responsibility for determining the contribution that mathematics can make to the educational development of the individual and then designing a program of evaluation sufficiently comprehensive to measure progress toward maximum benefit from all phases of this contribution. In order to measure this progress efficiently the techniques of measurement must be so flexible in nature that ready adaptation may be made to the characteristic differences that exist among individual abilities, in curriculum demands, and in guidance criteria. Such flexibility should in no sense interfere with the comprehensiveness of the program of evaluation. It is also important that the testing techniques be characterized by balance of emphasis between factual and functional objectives, between tangible and intangible outcomes, between the "how" and the "why," between mere recall and integrated thinking, and between measurement as a check on the completed process and as an aid to more effective instruction.

A satisfactory evaluation program should be further characterized by *continuity*. For efficient guidance there must be a continual check

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in Secondary Education, *Fifteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1940), p. 163.

[1] Hilda Taba, The Functions of Evaluation, *Childhood Education*, 15 (February, 1939), 245–246.

[2] The answer, given here, to this question is based largely upon that given by the Progressive Education Association, Mathematics in General Education, *Report of the Mathematics Committee of the Commission on the Secondary School Curriculum* (New York: Appleton-Century-Crofts, Inc., 1940), pp. 340–345.

on the student's progress, not only from the standpoint of immediate accomplishment but also from the standpoint of retention. Furthermore, the use of this check for prognostic and diagnostic purposes should be emphasized fully as much as its use as a measure of achievement.

If the evaluation program is to be thoroughly comprehensive and balanced, it must neglect no significant aspect of the subject matter covered and it must take into account all the important objectives which have been set up. Since it can command but a limited part of the school time, it must obviously consist of only a sampling of subject matter and problem situations. "The real task of evaluation, and the real purpose of testing, is to piece together the data of varied types and from many sources into a composite picture of the individual."[1]

However, unless care is exercised to ensure that the sampling is truly representative of the important aspects of the work, and reasonably balanced among these, the evaluation is likely to give an incomplete and distorted picture instead of a full and accurate one.

In the preparation of the instructional program the experienced well-trained teacher should enjoy a certain freedom from curriculum restraint. There should not be too much dictation as to the material to be covered in a specified period of time. This implies freedom also in the curricular materials and the testing techniques to be used in the evaluation program. On the other hand, the inexperienced teacher should seek the counsel of the supervisor or administrator in the setting up of objectives of instruction and in the selection of testing techniques and the construction of instruments for measuring student progress toward the attainment of such objectives.

The formulation of a sound philosophy of evaluation is but a necessary prerequisite to the construction of a satisfactory program of evaluation. With the above characteristics as guiding criteria the teacher or administrator may proceed more safely with the technical details incident to the selection or construction of valid and reliable instruments of measurement for use in any particular instructional situation.

Whether the problem is to provide an evaluation program from the point of view of the entire curriculum or of a specific subject matter field, there are at least five steps to be followed in setting up efficient testing techniques, *viz.*,

1. Determination of those significant aims and objectives which are to be the goals of instruction

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[1] Joint Commission, *op. cit.*, p. 165.

2. Provision of pertinent behavior situations to guarantee a valid measure of student reactions
3. Securing a reliable record of the student reactions
4. Accurate and systematic tabulation of the record as an aid in the deduction of implied results
5. Intelligent interpretation of the results in terms of student needs and as an aid to more effective instruction

Evaluation has a very definite place in the learning process which takes place in secondary mathematics. The program of evaluation should be designed in terms of the functional aims as well as of the factual aims of mathematical instruction. Is the instructional program such that functional learning and factual learning supplement each other? Are the students learning the "why" as well as the "how"? Are they building up integrated funds of information rather than stores of segregated bits of factual knowledge? Is the program of instruction such that it will provide the student with the techniques of critical thinking? Will it develop the ability (1) to distinguish between essential and unessential data, (2) to determine the reliability of facts and the reasonableness of results and conclusions, (3) to generalize circumspectly from known facts to unknown situations and new problems, (4) to evaluate arguments, ideas, and conclusions, critically? Carefully selected techniques of evaluation should be used in determining to what extent these aims have been realized by pupils, both as individuals and as groups. Furthermore, it should be constantly emphasized that the most significant functions of effective evaluation include not merely its use as an aid in determining pupils' marks but its use as an aid to the improvement of instruction.

The Techniques of Evaluation. The determination and perfection of techniques to be used in the evaluation of mathematical instruction is a definite responsibility of teachers of mathematics. These techniques, in the main, consist of teacher judgments and teacher-made or commercially produced tests. Teachers should be extremely conscientious in their efforts to evaluate student effort, and, in those situations which do not submit themselves very well to measurement scales, appraisals should be based on discriminative and impartial judgments arrived at after careful deliberation. Such judgments may be made through the medium of oral recitations in class, comparative class observations, the personal interview, the anecdotal record, and the prolonged case study.

If prepared tests are to be used, the teacher will at times have to consider the comparative advantages and disadvantages of standard-

ized[1] and teacher-made tests. Each possesses certain advantages over the other. The standardized test possesses norms which provide for more equitable comparisons between groups than can be made by teacher-made tests. They are usually constructed by individuals of wide experience and preparation in both subject matter and the techniques of testing. This increases the likelihood of greater reliability and validity. Standardized tests are usually subject to a greater degree of objectivity in administering and in scoring.

On the other hand, it is probable that standard tests which are used year after year may exert some "backward influence" which might partially nullify their validity so far as the content of the work of a particular class is concerned.

> It has also been found upon analysis that, on the average, pupils do well on material that has appeared in a number of previous examinations, although it was not included in the course of study, and that they do poorly with the material listed in the course of study but not included in previous tests.[2]

A distinct administrative advantage of the standardized test is that it diminishes the time which the teacher needs to devote to the details of a testing program. This, however, in the minds of some is a questionable advantage, the argument being that thoughtful effort on the part of the teacher in the details of test construction might make a distinct contribution in the direction of improved instruction. The use of such extramural tests as those prepared by the College Entrance Examination Board and the Board of Regents of the University of the State of New York, as well as some standardized tests designed primarily as final examinations, should be definitely restricted to the purposes for which they were designed and the situations to which they are related.

One of the major advantages of the teacher-made test over the

[1] See Maxie N. Woodring and Vera Sanford, "Enriched Teaching of Mathematics in the Junior and Senior High School" (New York: Bureau of Publications, Teachers College, Columbia University, 1938), pp. 31–40, 59–62, 71–76, 79–83. Also see the following yearbooks prepared by O. K. Buros: *Educational, Psychological, and Personality Tests of* 1933, 1934, *and* 1935 (1936), pp. 14–17, 40–41, 53; *Educational, Psychological, and Personality Tests of* 1936 (1937), pp. 10–12, 27–28, 35–36; *The Nineteen Thirty-eight Mental Measurements Yearbook* (1938), pp. 14–42, 83–84, 116–119; *The Nineteen Forty Mental Measurements Yearbook* (1940), pp. 268–314; *The Third Mental Measurements Yearbook* (1949), pp. 399–442. (New Brunswick, N.J.: Rutgers University Press.)

[2] Joint Commission, *op. cit.*, p. 168.

standard or extramural test is its flexibility and its adaptability to local situations and to repeated evaluations. Tests of subject-matter mastery should include the material which the class has studied and no other material. They should emphasize those things that have been emphasized in the class, and no item should have much place in a test for a particular class unless that class, in its study, has given some attention to that item or topic. It must be recognized that individual differences exist among classes and teachers just as they exist among individual students. Extramural tests can take some account of different levels of difficulty, but they cannot take account of differences in details of subject matter or emphasis, nor in the methods of presentation and the points of view of different teachers. Only the teachers themselves can make tests which will do this. Other advantages of teacher-made tests over standardized tests lie in their relative inexpensiveness and their inexhaustible availability.[1]

When the tests to be used are to be constructed by the individual teacher or by groups of teachers, there are two major problems to be considered, *viz.*, (1) What is to be tested? and (2) What is the most effective method of testing it? The answers to these two questions are to be found in the answers to certain supplementary questions. What are the instructional objectives to be measured? Is the test to be designed primarily for the purpose of measuring the attainment of standards, or is it to serve as a medium of instruction or as an aid in the educational guidance of the individual pupil? What are the distinguishing characteristics of prognostic, diagnostic, and achievement tests? Will factual questions, functional questions, or a combination of the two most adequately reflect the desired information? Such questions as these must be settled by the teacher before he will be able to construct an entirely satisfactory test.

Another question of special importance in this connection is whether an essay-type test or a new-type test is better suited to the particular situation. If the teacher is interested in having the test reflect something of the student's ability to organize and integrate information, then the essay-type test probably provides the better medium. There are certain types of mathematical subject matter that seem to limit themselves largely to the essay-type or problem-type of test, *e.g.*, solving verbal problems, solving geometrical originals, proving

[1] For a more detailed discussion of the comparative advantages and disadvantages of standardized and teacher-made tests, see C. W. Odell, "Traditional Examinations and New-type Tests" (New York: Appleton-Century-Crofts, Inc., 1928), pp. 19–31

theorems, constructing geometric figures, etc. The new-type tests, on the other hand, offer the opportunity of covering a wider range of material, and they are, in general, more objective. Such tests are likely to be more reliable than tests of the essay type, although the latter, if carefully constructed, may have a high degree of validity. New-type tests certainly give opportunity for a wider range of sampling, and for this reason they have certain advantages over essay-type tests in the matter of prognosis and diagnosis. For the same reason they make possible the inclusion of a more comprehensive range of items in measuring achievement. Their objectivity makes them easy to score and removes any personal element from the scoring, although the translation of scores into marks or grades may in some measure offset this advantage Aside from the fact that they are generally not good tests of organizing ability, their chief disadvantage lies in the fact that the construction of really good new-type tests requires much time, considerable experience, and great care [1]

New-type tests are made in various forms, and the determination of which of these forms is most suitable for a particular situation is sometimes a real problem. One must know and weigh the functions, advantages, and limitations of the different forms such as true-false, direct recall, multiple-response, completion, matching test, and other variations of the new-type test and must decide which form will lend itself most advantageously to the case in hand [2]

In any case, whether essay-type or new-type tests are being constructed, the following criteria should be observed with the utmost care [3]

1. *A test should be as highly objective as possible* The element of personal interpretation should be minimized in the determination of the correctness or incorrectness of student reactions to behavior situations.

[1] For more detailed discussion of the comparative advantages and disadvantages of essay-type and new-type tests, see Odell, *op cit*, pp 175–204

[2] For description of various forms of both essay-type and new-type tests, see H E Hawkes, E F Lindquist, and C R Mann, "The Construction and Use of Achievement Examinations" (Boston Houghton Mifflin Company, 1936), pp 125–159.

[3] This list has been constructed from suggestions made in H A. Greene and A. N Jorgensen, "The Use and Interpretation of Educational Tests" (New York: Longmans, Green & Co, Inc, 1929), pp 95–100, Hawkes, Lindquist and Mann, *op cit*, pp 17–159, 337–378; C. W. Knudsen, "Evaluation and Improvement of Teaching" (New York Doubleday & Company, Inc, 1932), pp. 327–333; Odell, *op cit*, pp 40–58

2. *A test should be reliable.* The reliability of a test is determined by the consistency with which it measures that which it does measure. There are many sources of unreliability, not all of which are attributes of the test itself. The behavior of the examiner, the mental and physical condition of the student, and the conditions under which the test is given have a great deal of influence upon the reliability of the results obtained from any test. Certain other causes of unreliability are inherent within the test itself, *e.g.*, ambiguity in the instructions for taking the test, lack of clearness in statement of problems and questions, inadequate sampling of the items of information to be tested, inefficient methods of scoring, and erroneous interpretations of test results.

3. *A test should be valid.* "If a test is valid, it is valid for a given purpose, with a given group of pupils, and is valid only to the degree that it accomplishes that specific purpose for that specific group."[1] Two significant attributes of validity are reliability and objectivity. However, their presence does not guarantee the validity of a test.[2] To be valid the test must be further characterized by that comprehensiveness and discriminative power most pertinent to the particular function for which it is designed. These criteria imply that the teacher must not only be thoroughly familiar with the objectives of instruction for the material to be tested but must also be well versed in the techniques of apt and precise phraseology and efficacious organization.

4. *A test should be economical of the teacher's time.* The amount of time required for the construction, administration, and interpretation of a test should not be excessive. The time element, however, is a function of the expected returns from the test.

5. *A test should be "student conscious."* The elements of the test should be couched in nonambiguous language, and reasonable tasks should be set for reasonable periods of time. In test items designed to measure understanding of a principle or ability to apply a principle, computation should be minimized.

6. *A test should motivate the best efforts of the students.* The questions should be so worded and presented that they will discourage guessing and bluffing. The use of the "catch question" should be minimized. However, occasional use of such questions might be justified from the point of view of stimulation of accurate thinking. A test should never be used as a means of punishment but should always tend to create in the mind of the student the attitude that it is worth while taking.

[1] Hawkes, Lindquist, and Mann, *op. cit.*, p. 21.

[2] Knudsen, *op. cit.*, pp. 329–330.

7. *A test designed to discriminate between students' abilities must provide for measurement of the entire range of abilities.* If anything like accurate discrimination between student abilities is to be approximated, there must be questions easy enough that all students can answer them and questions so difficult that perfect scores would be highly improbable, if not impossible. Some questions should be so designed that the student will have the responsibility of distinguishing between essential and nonessential data.

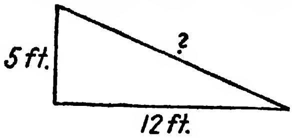
Prognosis and Guidance. As an aid to more effective pupil guidance, tests may be used to analyze present status of mastery and to predict possible future achievement. Such tests should be provided not only to measure mechanical ability and functional information, but also to make inquiry into students' interests, aptitudes, work habits, and study skills.

The inventory test is used for the purpose of "taking stock" of mathematical information and ability. It should show what a student knows about a certain topic. Under the modern philosophy of mathematical education the student has many opportunities to learn something of elementary algebra and a good deal of intuitive geometry by the time he enters the secondary school. As he proceeds up the instructional ladder, seasonal inventory tests carefully placed and skillfully used will prevent a great deal of unnecessary repetition of experience on the part of the student and waste of effort on the part of the teacher. They will also serve somewhat in the capacity of insurance against the monotony of learning which might result from student familiarity with teacher-selected material. Such tests may also be used effectively to bring to light the background which the students have for the study of new units and thus aid in the guidance program. The construction of an inventory test on a unit of instruction is not essentially different from the construction of a final achievement test on the same unit. The use of two such comparable tests, one before and the other after the teaching of the unit, will serve as a good indicator of the learning that takes place during the unit. Such a test on Exponents and Radicals is given here with its tabulation chart. There is also given the tabulation chart for a similar test, identical in form, which was given at the end of the unit. A comparative study of the two tables will give information concerning each pupil and the class as a whole on the learning situations recorded in the test.

TEST—EXPONENTS AND RADICALS

1. How many square roots does a number have?____________.
2. 5 is a square root of____because____ times____is____.

3. If $x = \sqrt{y}$, then____ $= y$. (Complete this statement.)

4. 

The hypotenuse of this triangle is____feet long.

5. Construct geometrically a line $\sqrt{3}$ inches long.

6. (a) $\sqrt{2}$ = ________; (b) $\sqrt{3}$ = ________; (c) $\sqrt{5}$ = ________

7. (a) $\sqrt{\frac{a^2}{b^2}}$ = ________; (b) $\left(\frac{4}{7}\right)^2$ = ________.

8. The simplest way to express $\sqrt{x^2y}$ is________.

9. In the expression $\sqrt[3]{x^2}$ the index is (a)_____ and the exponent is (b)_ The sign $\sqrt{\ }$ is called a (c)______sign.

10. $x^{3/2}$ means the (a)_____ root of the (b)_____ power of (c)______.

11. (a) ___; (b) $x^{3/4} \div x^{1/2}$ = ______; (c) $a^{3/2} \cdot a^{1}$ = ______.

12. (a) $x^5 \div x^5 = x^{(\ )}$. (Supply the exponent in the answer.) (b) To what *number* is the above answer equal?______.

13. (a) $a^5 \div a^7$ = ______. (b) Does a^{-3} equal $\frac{1}{a^3}$? ____.

14. $\sqrt{73}$ is between______and______. (Fill blanks with two consecutive whole numbers.)

TABLE 1. PUPILS' RESPONSES BEFORE STUDYING THE UNIT

Questions / Pupil	1	2	3	4	5	6			7		8	9			10			11			12		13		14	Pupil standing
						a	*b*	*c*	*a*	*b*		*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*a*	*b*		
A	x	x					x					x	x		x	x	x									8
B																										0
C		x				x	x					x	x					x						x	x	8
D		x										x	x		x										x	5
E	x	x																								2
F		x										x			x											3
G		x								x								x			x		x			5
H		x											x													2
Class standing	2	7	0	0	0	1	2	0	0	1	0	4	4	0	3	1	1	2	0	0	1	0	1	1	2	
Familiar material		x																								

TABLE 2. PUPILS' RESPONSES AFTER INSTRUCTION ON THE UNIT

Questions / Pupil	1	2	3	4	5	6			7		8	9			10			11			12		13		14	Pupil standing
						a	b	c	a	b		a	b	c	a	b	c	a	b	c	a	b	a	b		
A	x	x	x	x	x	x	x	x	x		x	x	x	x	x	x	x							x		17
B	x	x		x	x	x	x	x	x	x		x	x	x	x		x	x			x	x	x	x	x	20
C	x	x		x		x	x	x	x	x		x	x	x	x	x	x	x	x		x	x	x	x	x	21
D	x	x		x	x	x	x	x	x	x	x	x	x	x	x			x	x	x	x	x	x	x	x	22
E	x	x		x	x	x	x	x	x	x		x	x	x	x	x	x	x	x							17
F	x	x		x		x	x	x	x	x		x	x	x	x	x	x	x								15
G	x	x		x	x	x	x	x	x	x		x	x	x	x		x	x		x	x	x		x		19
H		x		x	x	x	x	x	x	x	x	x	x	x	x	x	x	x			x	x	x	x	x	21
Class standing	7	8	1	8	6	8	8	8	8	7	2	8	8	8	8	5	7	7	3	2	5	5	4	6	4	
Need to reteach			x								x								x	x			x		x	

An examination of Table 1 reveals the fact that, as a whole, this unit constituted new material for the entire class. The meaning of square root, item 2 of the test, was generally understood and no time needed to be spent on its development. Furthermore, the teacher was able to determine those pupils, *B*, *E*, *F*, and *H*, who would probably need special attention.

Table 2 gives the record of responses after the period of instruction on the unit. It reveals that pupils *A*, *E*, *F*, and *G* needed further individual attention on certain specific items and that, from the point of view of the entire class, items 3, 8, 11*b*, 11*c*, 13*a*, and 14 should be retaught.

A similar test that can be used effectively in connection with long-range planning in the junior high school is the

BUTLER TEST FOR MATHEMATICAL CONCEPTS

(Junior High School)

Directions. For each of the following statements or questions four choices are given. These are numbered (1), (2), (3), and (4). One of the four completes the statement or answers the question more correctly than any of

the others. Find the correct one in each case and *write its number* in the space at the right-hand edge of the page. Read the following example:

Example: The May reading of Mr. Jones' gas meter was 15,300 cu. ft. and the June reading was 15,600 cu. ft. The amount of gas used between the two readings was (1) 15,600 cu. ft.; (2) 300 cu. ft.; (3) 15,300 cu. ft.; (4) 600 cu. ft. (2)

The number 2 was put in the parentheses because the second answer (300 cu. ft.) was correct. Now do the rest of the examples the same way.

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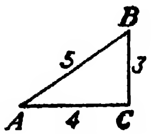
1. Which of the following expressions represents the ratio of two numbers? (1) $4 - 3$; (2) $6 + 7$; (3) $5/2$; (4) 8×5 . . . . ( )

2. Which of the following can be found only by indirect measurement? (1) the distance around a wagon wheel; (2) the weight of an iron ball; (3) the voltage of an electric current; (4) the distance from the earth to the moon . . . ( )

3. In this figure the tangent of angle A is:



(1) $5/3$; (2) $3/4$; (3) 20; (4) 3 . . . . ( )

4. Similarity (in its mathematical meaning) means: (1) a method used by bankers to find how much money is due on a loan; (2) solving a problem by a particular rule; (3) having the same shape; (4) being alike in every way . . ( )

5. A measurement is: (1) a ruler or yardstick; (2) the study of electric meters, water meters, etc.; (3) calculation of a distance by figuring it out; (4) finding out how many units in a certain amount . . . . ( )

6. The volume of a solid means: (1) how wide it is; (2) how many cubic units it contains; (3) its position; (4) its shape. . . . . ( )

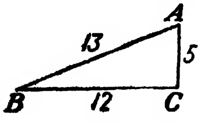
7. Which of the following is most nearly like a circle? (1) a half-dollar; (2) $7/8$; (3) a shoe box; (4) a baseball . . . . ( )

8. Which of the following objects is shaped like a rectangular prism? (1) a ball; (2) a brick; (3) a tomato can; (4) a triangular sheet of paper . ( )

9. A root of an equation is (1) a specified sum of money mentioned in the problem; (2) an answer that is not correct; (3) a value of the unknown quantity which makes the equation true; (4) a statement that two of the numbers are equal . . . . ( )

10. The four items listed below are about a coal pile, an airplane, a pencil, and a tree. Two properties of each are given. Read them all over and then decide in which case one of the properties depends upon the other: (1) the size of a coal pile and the number of tons it contains; (2) the number of an airplane and the speed at which it can travel; (3) the color of a pencil and its cost; (4) the age of a tree and the kind of tree it is . . . . ( )

11. In this figure the sine of angle A is:



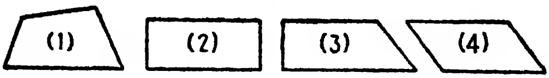
(1) $5/13$; (2) Angle C; (3) $12/13$; (4) $5 + 12 + 13$ . . . . . ( )

12. Which has the greater number of surfaces? (1) a marble; (2) a sheet of paper in the shape of a triangle; (3) a brick; (4) a long piece of wire. . . . . . ( )

13. Which of the following is a formula? (1) $A = \frac{1}{2}b \times h$;
(2)  (3) $11 = 5 + 6$; (4) 43%.............................( )

14. Pi (sometimes written π) means: (1) the name of an Italian coin; (2) the number of times the diameter of a circle can be divided into the circumference; (3) the amount which has to be paid annually on an insurance policy; (4) an algebraic number that can have different values according to the conditions of the problem in which it is used..........................( )

15. Which of the following figures is a trapezoid?



....( )

16. The average of any seven numbers is: (1) the middle number of the seven; (2) a number which is the sum of the seven numbers we started with; (3) a number which is obtained by adding all seven numbers together and dividing by seven; (4) the number which is obtained by multiplying all seven numbers together and dividing by seven..............................( )

17. Which of the following represents a pyramid? (1) a flat geometric figure with five straight sides; (2) 

(3) $7 + 3x = 13$; (4)  ....( )

18. A cone is: (1) one of the four chief methods of showing how numbers or quantities are related; (2) a geometrical object that has a square base and comes up to a point; (3) a word statement of a problem; (4) an object shaped about like the well-sharpened end of a pencil....................( )

19. Which of the following numbers is the fourth power of 3? (1) 12; (2) 64; (3) 81; (4) $\frac{4}{3}$..( )

20. If I buy a book for $2.00 and sell it for $1.85, the 15c difference represents my (1) investment; (2) interest; (3) loss; (4) profit..................( )

21. In the expression $4x^3 + 17y$, the coefficient of y is: (1) 3; (2) $17y$; (3) $4x^3$; (4) 17...( )

22. A statistical graph is: (1) a display of numerical facts by means of a sort of picture; (2) a method of calculating the distance between two points; (3) an arrangement of numbers in rows and columns; (4) a design made from geometrical figures..( )

23. The simple interest on $100.00 for two years at 5% is: (1) $5.00; (2) $4.00; (3) $10.00; (4) $2.00....................................( )

24. Taxation is a method of: (1) learning more about money; (2) raising money for public purposes; (3) getting immediate payment for goods sold; (4) figuring up the total expenses of a business..........................( )

25. A solution of an equation means: (1) the correct answer; (2) a formula; (3) two numbers which must be multiplied together; (4) the problem from which the equation comes..( )

26. A baseball has the general shape of: (1) a circle; (2) a quadrilateral; (3) a sphere; (4) a perimeter...( )

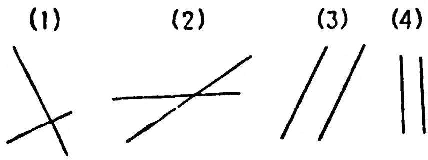
27. Which of the following reasons best explains why algebra has been called "a tool of science"? (1) algebra is a science itself; (2) most people who study algebra become interested in science; (3) the use of algebra makes it easier to state scientific laws and to work with them; (4) science is taught in laboratories, while algebra is not..... . ... (

28. Read this problem carefully: "Find the distance covered in any given number of hours by a train moving at 40 miles per hour." This is a problem involving: (1) quadratic equations; (2) equivalent fractions; (3) direct variation; (4) metric measurement.................................. (

29. In the expression $7^2 - 9 = 40$, the exponent of 7 is: (1) 2; (2) 9; (3) 40; (4) 49.. . . .. (

30. The thing we use to show numerical facts by means of a picture is called: (1) a table of statistics; (2) a graph; (3) a logarithm; (4) an equation. (

31. Which of the following pairs of lines are perpendicular to each other? (



32. Which of the following expressions is a proportion? (1) 51 %; (2) $\frac{3}{4} = \frac{9}{12}$; (3) $7 \times 82 =$; (4) $13 + 6 - 2$...... . . .. (

33. Insurance is: (1) financial protection against loss; (2) a business organization; (3) the payment of a certain sum of money every month or year; (4) loss of money by fire or accident...................... .... .. (

34. If I buy a book for $1.00 and sell it for $1.35, my profit is (1) $1.00; (2) $1.35; (3) $2.35; (4) 35c... .. .. (

35. Congruence means: (1) the comparison of measurements in the metric and the English systems; (2) the correctness of an estimate (for the answer to a problem); (3) a comparison of the sizes of two angles; (4) being exactly alike (in the case of two or more geometric figures)................. . .(

36. Which of the following expressions is an equation? (1) 4×3; (2) $18 - 6 + 4 + 11 - 7$; (3) $12\frac{1}{2} : 1$; (4) $6 + 11 = 17$....... .. .. (

37. If positive numbers represent miles of travel eastward, then negative numbers represent: (1) hours spent in travel eastward; (2) miles of travel southward; (3) miles of travel westward; (4) miles of travel northward (

38. If I loan $100 at 6% compound interest for ten years, the amount which will draw interest the second year is: (1) $106; (2) $100; (3) $6; (4) $94(

39. Which of the following expressions is an algebraic fraction?

(1) $\frac{5}{3}$; (2) $\sqrt{\frac{3}{7}}$; (3) $\frac{2a}{3}$; (4) $\frac{15}{12} \div \frac{6}{7}$ ... .(

40. Which of the following objects is shaped like a cylinder? (1) a ball; (2) a brick; (3) a tomato can; (4) an automobile tire.. .... (

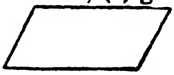
41. Which of the following figures is a triangle?

.. .. ( )

42. Graphic representation means: (1) dishonest business methods; (2) investing money so that it will draw interest; (3) showing mathematical facts by means of pictures; (4) working problems by means of calculating machines . . . . ( )

43. Sometimes letters are used for numbers, and sometimes letters and numbers have different values according as they have plus signs or minus signs in front of them. We call such letters and numbers: (1) integers; (2) algebraic numbers; (3) positive numbers; (4) geometrical numbers . . . . ( )

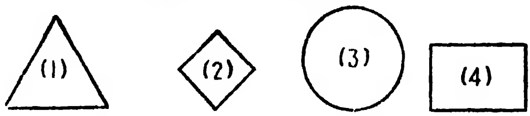
44. The name of this figure



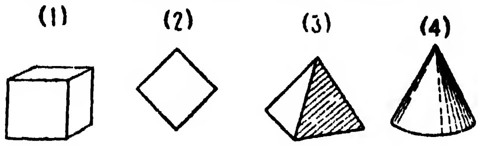
is: (1) rectangle; (2) triangle; (3) perpendicular; (4) parallelogram . . . . ( )

45. An algebraic product is: (1) the result we get when two or more algebraic numbers are multiplied together; (2) the general method of solving an equation; (3) the answer we get when we add one algebraic number to another; (4) one algebraic number subtracted from another . . . . ( )

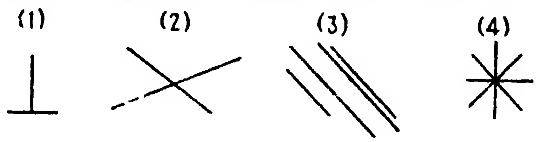
46. Which of the following figures is a square? . . . . ( )



47. Which of the following figures looks most like a cube? . . . . ( )



48. Which of the following sets of lines are parallel lines? . . . . ( )



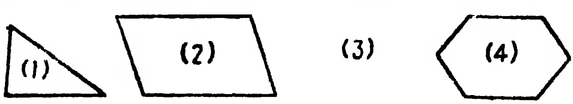
49. An algebraic factor is: (1) any algebraic number; (2) two or more algebraic numbers multiplied together; (3) an algebraic number added to some other number; (4) an algebraic number which is multiplied by some other number . . . . ( )

50. The length of an object might be measured in: (1) pounds; (2) kilowatts; (3) seconds; (4) inches . . . . ( )

51. The mathematical meaning of equality is: (1) the correct answer for a problem; (2) two or more things having the same value or containing the same number of units of the same kind; (3) a system of measurement used by the British and American people; (4) a problem which may be solved by either of two different methods . . . . ( )

52. The fifth root of 32 is: (1) $\frac{32}{5}$; (2) 5×32; (3) 8; (4) 2 . . . . ( )

53. Which of the following figures is a rectangle? . . . . ( )



54. An algebraic expression containing the second power but no higher power of the unknown quantity is called. (1) a linear equation, (2) a quadratic, (3) a radical, (4) a logarithm ( )

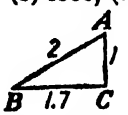
55. A map of your town would be: (1) a statistical graph; (2) a drawing to scale; (3) a perspective drawing, (4) a three-dimensional picture ( )

56. The distance around a flat figure is called its (1) area, (2) perimeter; (3) cross section, (4) diameter . ( )

57. The following list contains three examples of measurement that are familiar to most people These are: the length of a radio wave, the weight of a load of coal, and the length of a minute as determined by the United States Naval Observatory instruments. *HOW MANY* of these measurements are absolutely exact (that is, how many involve no error whatever)? (1) all three, (2) two, (3) one, (4) none ( )

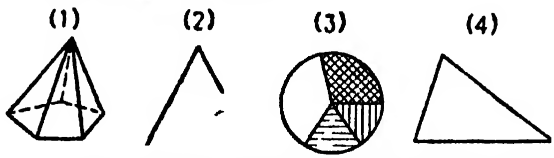
58. A merchant buys an article for $6.00 He marks it to sell for $8.00 but agrees to reduce this to $7.50 if the purchaser pays cash This amount ($7.50) is known as (1) the list price, (2) the net price, (3) the discount, (4) the rate of exchange ( )

59 Which of the following is a measure of area? (1) square yard, (2) gallon; (3) foot, (4) cubic inch ( )

60. 

In this figure the cosine of angle A is. (1) angle B, (2) $\frac{1}{2}$, (3) a right angle (4) 1.7×2 ( )

61. Which of the following is an angle? . . ( )



62. A point where two or more lines meet is called· (1) an axis, (2) a base, (3) a vertex, (4) a side ( )

63. An "approximation in a measurement" means. (1) the thing that is being measured, (2) the instrument that is used to make the measurement, (3) applying the result of the measurement in working out a problem, (4) a result that is not exactly correct .... ( )

In the prediction of mathematical achievement some of the most important factors seem to be comprehension of general mathematical techniques, classroom attentiveness, originality, habits of study, and general intelligence. The most efficient prediction seems to be accomplished through a combined use of prognostic tests, intelligence tests, and teachers' marks. Also personality ratings have been found to be very effective in the prediction of success in freshman engineering mathematics.[1]

As a corollary to their use as an aid in prediction of achievement,

[1] R. D Perry, Prediction Equations for Success in College Mathematics, *Contribution to Education* 122 (Nashville, Tenn., George Peabody College for Teachers, 1934).

prognostic tests can be used to reduce the number of failures either by eliminating those who are unprepared, or unable for any cause, to proceed further with mathematical study or by providing a basis for the construction of a differentiated mathematical curriculum. Such tests should also serve as an aid in the vocational and educational guidance of pupils and in the better classification of pupils. The discovery of superior ability and unusual aptitude in mathematics is just as important a function of prognosis as is the discovery of the inferior or average.

For the construction of efficient prognostic tests in mathematics the teacher should be familiar with those abilities and interests essential to further progress. Mathematical tests which are to be used as an aid in vocational guidance should be based on a knowledge of those mathematical skills, concepts, and principles incident to success in any chosen vocation. The general characteristics of comprehensiveness, discriminative power, reliability, validity, balance, and flexibility must then be carefully observed in the framing and organization of the test items.

Diagnosis and Remedial Teaching. Probably one of the most significant steps that has been taken in recent years toward improved instruction is that of incorporating into the instructional program plans for discovering learning difficulties and detecting needs for remedial teaching. Such plans call for the intelligent use of inventory and diagnostic tests along with personal interviews to discover and analyze pupil difficulties with a view to setting up specific remedial measures to correct errors and remove difficulties. The characteristics of an efficient program of diagnosis may be summarized as follows:

1. Such diagnosis must be made in connection with worthy objectives of a good educational program.
2. It must be objective, reliable, and valid.
3. It should be as specific as the desired outcomes permit and as the possibility of localization of symptoms allows within the limitations of practicality.
4. It should yield results that would be comparable over a period of time and between groups of students.
5. It should be sufficiently precise to note progress during small units of time.
6. It should be comprehensive.
7. It should be appropriate to the educational program.
8. The person making the diagnosis must understand the educational program and be familiar with the fundamental problems of children.[1]

[1] Ralph W. Tyler, Characteristics of Satisfactory Diagnosis, *Thirty-fourth Yearbook of the National Society for the Study of Education* (Bloomington, Ill.: Public School Publishing Company, 1935), Part II, Chap. 6, pp. 95–111. Quoted by permission of the Society.

An illustration of how a test may be used for both class and individual diagnosis is shown in the following test and its accompanying tabulation chart. The test was used in a class of 17 ninth-grade students.

Test

NAME.................... DATE.. ....

Solving Equations and Problems

Part I

Solve the following equations for the value of the unknown terms:

$a + 9 = 29$
$a =$

$m + 35 = 47$
$m =$

$5x + 7 = 27$
$x =$

$2y + \frac{14}{3} = 8\frac{2}{3}$
$y =$

$6x + \frac{1}{2} = 30\frac{1}{2}$
$x =$

$12x + .7 = 24.7$
$x =$

$2.5r + 4 = 129$
$=$

$3x - 8 = 7$
$x =$

$5y - 24 = 26$
$y =$

$9x - 5x + 62 = 116$
$x =$

$7x + 3 = 4x + 12$
$x =$

$11x - 6 = x + 4$
$x =$

$\frac{x}{2} - \frac{x}{4} = 3$
$x =$

$2b + \frac{b}{3} = 22$
$b =$

Part II

1. Two numbers differ by 7. The smaller is s. Express the larger.
2. If $\frac{4}{5}$ of a number is decreased by 6, the result is 10. Find the number.
3. Find two consecutive odd numbers whose sum is 204.
4. Find the side of an equilateral triangle if the perimeter is 36 inches.
5. The length of a field is three times its width and the distance around the field is 200 rods. If the field is rectangular, what are the dimensions?
6. A and B own a house worth \$16,000 and A has invested twice as much capital as B. How much has each invested?

In Table 3 the totals of each column are for the entire class of 17 pupils. The tabulation of errors for only five pupils is given to illustrate individual diagnosis. From the totals listed in the respective columns, there can be obtained a rough estimate of the efficiency of the class on particular types of problems as well as an indication of the places where the class was having trouble. For example, the third column gives the information concerning equations with fractions which call for a simple subtraction in obtaining the solution. There were two such problems on the test; hence there were in all 34, 2 $\times$ 17,

chances for error. Actually only nine errors occurred; consequently the class efficiency on this type of problem was 73 per cent. Equations with fractions and verbal problems are seen to have been the trouble spots for the class.

The difficulties of each individual pupil are found by examining the horizontal rows. The chart shows that B. E. understood the work covered by the examination and that she was accurate in the fundamentals of arithmetic. B. E. did not need any remedial work and

TABLE 3. DIAGNOSIS CHART

Name of student	Equation and simple subtraction 2	Same, coefficient 1	Same, common fraction 2	Same, decimal fraction 2	Equation and simple addition 2	Double transposing 3	Equation common fraction 2	Verbal	Grade	Class standing
L.M....................	..	..	..	..	..	1	2	5	60	12
G.S....................	..	..	2	..	..	..	..	..	90	4
J.S....................	..	..	..	..	..	1	..	6	65	11
C.B....................	2	1	2	2	2	2	2	6	5	17
B.E....................	..	..	..	..	..	..	..	..	100	1
Total number of errors....	3	1	9	7	8	8	18	46		
Number of problems to solve..................	34	17	34	34	34	51	34	102		
Number correct.	31	16	25	27	26	43	16	56		
Per cent correct.	91	94	73	80	80	80	47	55		

could work on in advance of the class and try her skill on something new. J. S. made one mistake due to a sign in the type of problem charted in column six. Since she worked the other problems correctly, this was probably accidental. However, she was unable to translate any of the verbal problems into proper symbols. J. S. therefore had to have more experience in translation and did not need to concern herself very much with solution of mechanical problems. C. B. was absent a great deal during this period, and her chart shows that she knew practically nothing about the unit. She required a great deal of attention during the remedial work.[1]

The scientific use of such a program of diagnosis is an important

[1] Peter L. Spencer, Informal Tests for Diagnosis and Remedial Teaching, *The Mathematics Teacher*, **16** (1923), 175–182.

aspect of functional teaching. Its real value will be definitely dependent upon a follow-up program of remedial teaching and a careful check on and interpretation of attained results. To be most effective the remedial material should possess the following characteristics:

1. It should be selected to bring about certain definite ends.
2. It should be on component elementary skills.
3. It should be largely capable of self-administration by the student.
4. It should be of such a nature that it could be administered to a class, to an individual, or to a group.
5. It should be correlated with the instructional material being used.
6. It should be provided with answers.[1]

Self-diagnosis by Students. One of the most important functions of tests as an aid to the improvement of instruction is their use by the students to secure evidence concerning their individual development. Such tests are called "practice tests" and may be either oral or written. Tests of this type can play a very important role in the assimilative period of instruction. They can aid the student in self-diagnosis and should never be used by the teacher in any other capacity than to help the student discover for himself information concerning his status of achievement in intelligent understanding of subject matter, the speed and accuracy with which he can perform the prescribed operations, and his relative progress as a member of his class group. Such tests must be shaped to reflect individual efficiency in the perspective of group activity.

Oral tests may be used with material that calls for responses which can be readily obtained and simply stated. They may be administered through pointed questions promiscuously, yet evenly, distributed over the entire class or through the medium of team contests. The two principal key-notes of successful oral practice are speed and accuracy of response.

Written tests may be used for both the simple-response and the difficult-response type of practice. As in the case of the oral tests, written tests may be shaped to emphasize speed and accuracy. It should be reemphasized at this point, however, that understanding is a major responsibility of instruction in secondary mathematics and some of the practice tests should be designed to this end.

Some of the different forms for administering written practice tests are: (1) all students at board, (2) some at the board and others in their

[1] L. H. Whitcraft, Remedial Work in High School Mathematics, *The Mathematics Teacher*, **23** (1930), 36–51.

seats, (3) all students in their seats. In any of the above cases all students may be working on the same assigned problems, or separate groups may be working on separate problems. These problems may be dictated by the teacher or may be printed, mimeographed, or otherwise reproduced. Timed tests frequently serve to stimulate interest and attention through competition with other students or competition with one's own previous time record on similar material. Precautions which the teacher must observe are as follows: (1) Do not overemphasize speed at the expense of accuracy; (2) provide for check-up and practice on understanding as well as on speed and accuracy; (3) vary the type of practice material to prevent monotony of effort; and (4) do not continue practice to the point of fatigue.

Achievement. In the measure of achievement the teacher is not merely interested in testing mechanical proficiency in certain fundamental processes and factual information. He is also interested in the measure of reasoned understanding of concepts, techniques, and principles. Such a testing program should be so designed that it will compare and discriminate between relative abilities as well as measure retention and understanding of learning. The selection of test items must be in terms of an authoritative list of ultimate objectives, which in turn have been carefully analyzed into immediate aims of instruction in the light of the significant implications and limitations of the local situation. Although comprehensiveness is one of the important characteristics of an efficient achievement test, it is quite obvious that no such test can be sufficiently comprehensive to include all items of specific information into which an instructional unit may be analyzed. The test is, thus, a function of the different items used. They must therefore represent an adequate sampling of the entire unit, and "the worth or effectiveness of [each] item depends . . . not only upon its desirability for inclusion in the curriculum and upon its 'difficulty,' but also upon its power to discriminate between pupils of high and low levels of general achievement."[1] The validity of any achievement test is largely a function of the validity of each individual test item. In the construction of an achievement test for a specific instructional unit and for a given group of pupils the selection of content should be subjected to the following restrictions:

1. If we think of the entire group of pupils as separated into a number of levels of general achievement, then for each of these levels the test must

[1] Ben D. Wood, E. F. Lindquist, H R Anderson, Educational Tests and Their Uses: Basic Considerations, *Review of Educational Research*, 3 (1933), 16.

contain an approximately equal number of items calling for information that the pupils at that level *have* learned; or for ideas and generalizations that they *do* understand; or for judgments, applications, or reasoning of which pupils at that level *are* capable.

2. The items thus selected with reference to each level of achievement must *discriminate* as sharply as possible between pupils above and below that level; that is, it must (ideally) be highly probable that all pupils above that level will succeed on each of those items and that all pupils below that level will fail on each of them. (This is particularly important in recognition tests with reference to items that call for judgments beyond the ability of the pupils tested.)

3. The items must be such that the response scored as the "correct" or "best" response would be considered so by competent authorities. . . .

4. The items must hold the pupil responsible for understandings, abilities, or information that it is believed will contribute to the realization of the objectives of instruction; that is, they must be based upon subject matter which has been *authoritatively* and *specifically* selected and described for purposes of instruction and which does belong to the field of subject matter involved.[1]

Interpretation and Use of Results. The value of any program of evaluation is very definitely dependent upon the interpretation and use of the results obtained. It is highly important that the results of a given test be interpreted and used in the context of the function for which the test was constructed and administered. For example, the results from a test designed solely for diagnostic purposes should never be used for the purpose of measuring achievement and then assigning grades. There are sometimes fundamental differences in the construction of such tests as well as the moral obligation of the teacher to play fair with the student.

In many cases it is necessary that the teacher be familiar with certain simple but fundamental statistical procedures in order to derive maximum benefit from a testing program. The techniques of classification and tabulation of data; grouping into significant class-intervals; determining range of distribution; computation of measures of central tendency, variability, and relationship; and the ranking of scores are the more important statistical measures with which the teacher should be familiar in order to summarize efficiently the results of a testing program.

Not only must the teacher have a certain amount of mechanical proficiency in the techniques for tabulating and analyzing test data, but also he must be able to make logical inferences from such findings. He must know the appropriateness of various measures and the extent

[1] *Ibid.*, p. 19.

and limitations of their implications. The conclusions drawn must be consistent with the fundamental assumptions underlying the tests and the statistical measures used.

In the measure of achievement the teacher must know the difference between a test score and a grade. Furthermore, he must know when test scores should be translated into grades and be familiar with the recommended techniques for careful conversion.[1] He must know how to detect a typical class error from a diagnosis test and must be able to determine whether particular detected errors imply the need for group or individual remedial measures.

Teachers should be familiar with the various purposes and the limitations of each aspect of a functional testing program. For maximum value the results should be used within the domain they are designed to serve, in the effort to construct from all sources of relevant information a composite, yet comprehensive, picture of the individual student.

Exercises

1. Distinguish between diagnostic, prognostic, and achievement tests as to characteristics and function

2. Name different ways in which each type of test may be used in the improvement of instruction in mathematics

3. How specific should one be in the formulation of the objectives for testing?

4. What are some of the recommended measures for improvement of essay-type examinations?

5. Contrast the relative effectiveness of different kinds of new-type tests (true-false, multiple-choice, completion, etc).

6. Discuss the relative merits of factual and functional testing in secondary mathematics

7. What are some recommended procedures for measuring mathematical aptitude?

8. What is meant by the validity of a test?

9. What is meant by the reliability of a test?

10. Give examples of material from secondary mathematics which you consider not well adapted to new-type tests.

11. In test construction how much importance should be given to "range of difficulty" and "distribution of item difficulty"?

12. Briefly evaluate the advantages and disadvantages of extramural tests.

13. What are some of the common errors in secondary mathematics that call for remedial work?

14. What are some of the more important techniques for discovering pupil errors?

15. What are some of the recommended techniques for determining the degree of difficulty of test items?

[1] Knudsen, *op cit*, pp 355 366.

ULLSVIK, BJARNE R.: An Attempt to Measure Critical Judgment, *School Science and Mathematics*, **49** (1949), 445-452.

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CHAPTER IX

THE PROFESSIONAL PREPARATION OF TEACHERS OF SECONDARY MATHEMATICS[1]

In the educational crisis that has arisen during recent years, mathematics, as an integral part of the curriculum of the secondary school, has found the voice of criticism rather severe. The justification for such vehemence is not in the shortcomings of the subject, one which through the ages has been a beacon light to scientific discovery and intellectual progress; rather, such justification is to be found in the poor instruction imparted by unprepared and nonenthusiastic teachers. It is poor teaching that leaves the impression that mathematics is *merely* a tool subject composed of a conglomerate mass of signs, symbols, and laws of operation. Undefined standards of preparation and varied patterns of certification have contributed very materially to the general chaotic condition that exists. No general agreement seems to be in evidence as to what constitutes adequate preparation for the teaching of secondary mathematics.[2] As long as this situation exists, the employment of mathematics teachers, the assignment of mathematics classes, and even the inclusion of mathematics in the educational program will remain on a rather unstable professional basis. There is great need for the studious definition of a program designed to prepare prospective teachers for a true profession of teaching of mathematics.

The Professional Preparation of Teachers of Secondary Mathematics. There are two equally important aspects of any true profession, *viz.*, significant knowledge and effective technique. One can-

[1] Portions of this chapter originally appeared in *The Mathematics Teacher*, **32** (1939), 99-105, under the title "The Professional Preparation of Mathematics Teachers."

[2] H. T. Karnes, "The Professional Preparation of Teachers of Secondary Mathematics," unpublished Ph.D. dissertation (Nashville, Tenn.: George Peabody College for Teachers, 1940), pp. 65-124.

National Survey of the Education of Teachers, *Bulletin* 10, Office of Education, 1933 (Washington: Government Printing Office, 1935), **3**, 17-48, 141-142; **5**, 300-307.

Report of the Committee on the Subject Matter Preparation of Secondary School Teachers, *North Central Association Quarterly*, **12** (1938), 471-477.

not be efficiently professional if there is any distinct weakness in either aspect. A truly functional program of professional preparation must therefore place emphasis on the acquiring of knowledge significant to the chosen profession and also on the acquaintance with and use of the more efficient techniques of that profession.

Such studies as the National Survey of the Education of Teachers[1] and the Report of the Committee of the North Central Association on "Subject Matter Preparation of Secondary School Teachers"[2] tell us that

> . . . changes in the nature of the high school student body, the expansion and diversification of the program of studies, the new responsibilities for guidance, the incorporation of student activities into the curriculum itself, the tendency toward curricular integration, the marked changes in educational objectives, all together indicate a revolution in secondary education to which the subject matter preparation of teachers has certainly not been adjusted with sufficient rapidity or appropriateness.[3]

The Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics, as well as certain more localized groups, have made notable efforts through their committees and publications to improve the situation as far as mathematics is concerned. The interest of these groups has been directed to the better preparation of teachers as well as to the improvement of the curriculum. The fact remains, however, that the teaching of secondary mathematics in the United States is hardly an established profession.[4] There are individuals teaching secondary mathematics whose academic preparation includes neither a major nor a minor in mathematics. They have no great interest in the subject and, in their teaching, can do no more than treat it in a superficial and fragmentary way. They have little appreciation of the values of mathematics, of the role it has played in the evolution of our civilization, or of its possibilities for integrated development. Likewise, there are many teachers of mathematics whose preparation has been adequate in the field of mathematics itself, but who are unhappy and inefficient in their work because they are required to teach one or more classes in other subjects. Their programs of preparation should have been broad enough to enable them to teach in at least one additional field. There are still other teachers who know

[1] National Survey of the Education of Teachers, *op. cit.*, Vol. I–VI.

[2] Report of the Committee on the Subject Matter Preparation of Secondary School Teachers, *op. cit.*, pp. 439–539.

[3] *Ibid.*, p. 440.

[4] Karnes, *op. cit.*, pp. 125–178.

their subject well but who, because of their lack of patience with student difficulties and their unwillingness to adjust their teaching to the varying abilities of different students, actually destroy the interest of many students in mathematics when, under more favorable conditions, it might have been made to flourish. Such teachers lack that professional attitude which should impel them to discard any sense of intellectual superiority and to view the subject through the eyes of the immature student so that they might patiently guide and encourage him in his efforts and stimulate his interest in further exploration of the field of mathematics.

Masterful scholarship in a body of relevant knowledge is an absolute essential for effective teaching, but it must be supplemented by a proficiency in the use of efficient techniques of instruction. Neither should be emphasized to the exclusion of the other, but a proper balance should be maintained throughout the preparation program. We do not want teachers of mathematics to be "teachers who have nothing to teach," neither do we want them to be "mere purveyors of knowledge and promoters of skill."

Significant Knowledge. There is a trichotomy of knowledge significant to the teacher of mathematics which might be classified under the headings of general knowledge, professional knowledge, and specialized knowledge. We are living in an age in which events take place very rapidly. This rapidity of development and its implications for future change tend not only to stagger the imagination but also to encourage a satisfaction in superficiality of information. Things happen too fast for any individual to be able to attempt a very thorough and systematic acquaintance with the fundamentals of all lines of development. We have to be satisfied with a type of superficial information along some lines. It is for such reasons as this that the teacher of mathematics should have a broad educational background against which to project his thinking and in which to orient his appreciation of values. Such an informational background should be related to "the major areas of human experience" and designed to build up a more intelligent understanding of the part mathematics has played in the evolution of modern civilization and a deeper appreciation of its relation to social progress. Such "a program of general education for prospective teachers should acquaint them with the various institutions and forces that influence modern life and with the contributions that the major fields of learning have made and are making today to the progress of civilization."[1]

[1] E. W. Knight, Discussion: Directors of Needed Improvement in Subject

A professional attitude should be a *sine qua non* for every teacher of mathematics. The term "professional attitude" is interpreted here to mean an enthusiastic interest in mathematics as a chosen field of study and service, an inspired concept of the value of mathematics in the structure of civilization, and an eager readiness to interpret carefully and thoughtfully those fundamental laws, mechanical processes, generalizing procedures, and possibilities of practical applications which so definitely characterize mathematics as a field of study and endeavor. In addition to providing a general education for cultural background, the program for the professional preparation of teachers of mathematics should equip such a teacher with an integrated philosophy of education, a devotion to teaching as a profession, and a sense of responsibility for the contributions he will be expected to make in his chosen field of work. This body of professional knowledge should be provided through courses designed to acquaint the individual with the place and function of education in our social order, the interrelationships that exist between the various professions, and the manifold opportunities for service which present themselves to teachers; to build up "sympathetic understanding of the mental, physical, and social characteristics of the children or adults to be taught;"[1] and to provide "opportunities for acquiring a 'safety minimum of teaching skill' through observation, participation, and actual practice under supervision."[2]

Although the cultural background and the body of professional knowledge are essential elements to the program of professional preparation of teachers of mathematics, such a program must not overlook the fact that sound scholarship is a fundamental qualification of the teacher. This scholarship, however, should be *relevant* to the problem of teaching. From the point of view of the teacher of mathematics, what materials provide opportunity for development of *relevant* scholarship? The scholarship that is of service in the advancement of mathematical thought and research is not, in every case, the same as that which can be of service in the education of adolescents and immature thinkers. It is almost proverbial to state that "he who

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Matter, *Twenty-sixth Yearbook of the National Society of College Teachers of Education* (Chicago: University of Chicago Press, 1938), p. 16.

[1] E. S. Evenden, Summary and Interpretation of the National Survey of the Education of Teachers, *Bulletin* 10, Office of Education, 1933 (Washington: Government Printing Office, 1935), 6, 93.

[2] *Ibid.*, p. 94.

learns that he may know and he that learns that he may teach are standing in quite different mental attitudes."[1]

The specialist in mathematics has need of a synthetic type of scholarship in which he seeks mastery not only of the fundamentals of mathematical thought but also of a closely interwoven chain of logic and of methods of making deductions and implications. His only need for the analytic type of scholarship is as a tool to be used in the aid of mathematical research. He is not primarily concerned with the questions of the place of mathematics in the educational program or of the practical value of mathematics to the average man. He is concerned with mathematical implications rather than with educational implications or practical applications of mathematics except, possibly, by the actuary, engineer, or other professional users of applied mathematics. He is the producer of mathematics.

On the other hand, the teacher of mathematics is the seller of mathematics. It is he who must convince the consumer of the value of his subject and, through the medium of efficient service, secure and retain consumers. He is constantly confronted with questions as to the value of mathematics as an asset to the individual and as a significant element in the program of general education. His interest in mathematics must embrace its educational implications and practical applications as well as its intrinsic subject appeal. Thus the preparation of the teacher of mathematics should emphasize that type of scholarship which seeks to integrate the subject with broad fields of learning and to relate it to general human activity and interest. The teacher must learn to evaluate mathematics in the light of its role in the history of civilization, its contribution to the present social order, and its relation to future progress.

Furthermore, since it is to be his responsibility to assist immature learners in the mastery of mathematics, the prospective teacher of mathematics should not only strive for proficient mastery of the subject, but he should also make every effort to be conscious of the processes by which he arrives at that mastery. He should pause at significant points for moments of reflection in which he should attempt to analyze the learning processes involved and to evaluate the materials studied. Competent scholarship, which emphasizes understanding and mastery of fundamentals, must be constantly emphasized. These

[1] H. S. Tarbell, Report of the Sub-committee on the Training of Teachers, *Proceedings of the National Education Association* (Washington: National Education Association, 1895), p. 240.

fundamentals will vary somewhat, according to whether the prospective teacher expects to teach in the elementary school, junior high school, senior high school, or junior college. To supplement this body of minimum essentials every teacher of mathematics should be encouraged to acquire a certain synthetic proficiency in some chosen line of mathematical endeavor to serve as a reserve of information which he might frequently use as an aid to individual exploration in the unknown realms of mathematical knowledge or in the expanding domain of significant applications of mathematical principles and techniques.

Relevant Scholarship for the Teacher of Secondary Mathematics.[1] The teacher of secondary mathematics should have some appreciation of the part that mathematics has played throughout the centuries of progress. Furthermore, he should have those contacts with the subject matter and history of mathematics that would enable him to formulate an intelligent notion of the meanings of mathematics. In 1901 Bertrand Russell defined mathematics in words that are superficially facetious but fundamentally significant when he said: "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."[2] In discussing this definition Bell says that it has four great merits: (1) it shocks the self-conceit out of common sense, (2) it emphasizes the entirely abstract character of mathematics, (3) it reduces all mathematics and the more mature sciences to postulational forms, and (4) it administers a resounding parting salute to the doddering tradition that mathematics is the science of number, quantity, and measurement.[3] Mario Pieri is responsible for the statement that "mathematics is an hypothetico-deductive system" which merely means that mathematics is a system of logical processes whereby conclusions are deduced from whatever fundamental assumptions and definitions there may be hypothesized. To Benjamin Peirce goes the credit for expressing these thoughts in the more explicit form: "mathematics is the science which draws necessary conclusions."[4] To think mathematically is to free oneself by abstraction from any peculiarity of subject matter and to make inferences and deductions justified by fundamental premises.

[1] Portions of this section originally appeared in *The Kadelpian Review*, **14** (1934), 23–27, under the title "The Scholarly Teacher of Mathematics."

[2] Bertrand Russell, Recent Work on the Principles of Mathematics, *International Monthly*, **4** (1901), 84.

[3] E. T. Bell: "The Queen of the Sciences" (Baltimore: The Williams & Wilkins Company, 1931), pp. 16–17.

[4] Benjamin Peirce, "Linear Associative Algebra," Sec. 1, lithographed, 1870. Reprinted in the *American Journal of Mathematics*, **4** (1881), 97.

There are four significant methods of mathematics which might be stated as follows:

1. Scientific, leading to generalizations of widening scope
2. Intuitive, leading to an insight into subtler depths
3. Deductive, leading to a permanent statement and rigorous form
4. Inventive, leading to the ideal element, and creation of new realms[1]

The competent teacher of mathematics should realize that in any system of constructive thought the validity of the conclusions rests entirely upon the validity and consistency of the assumptions and definitions upon which the conclusions are based. It is important that students should have this point of view, and the teacher should make every reasonable effort to assist them in acquiring it.[2] From a logical point of view the set of fundamental assumptions should be consistent, independent, and categorical, but pedagogically the only essential requirements are that they be consistent and categorical. An appreciation of this dependence upon fundamental assumptions and definitions and previously established theorems should help to develop an "if-then" mental attitude which should function in a more intelligent interpretation of human events. A student in the secondary school is by no means to be taught the philosophical and logical aspects of non-Euclidean geometry, but the teacher should be familiar with the high points in the history of Euclid's famous fifth postulate. He should know of the efforts of mathematicians to demonstrate the dependence of this parallel postulate upon the remaining postulates of Euclid's fundamental set. He should appreciate the fact that Lobatschevskian and Riemannian geometries are just as logically sound as the commonly accepted Euclidean geometry; that, although the Euclidean postulates are accepted because they seem to conform more nearly to the everyday experiences of the world in which we live, in Poincaré's world of changing temperatures the parallel postulate would be absurd.[3] The teacher so informed would have a significant appreciation of geometry as a form of postulational thinking and thus be better

[1] J. B. Shaw, "Lectures on the Philosophy of Mathematics" (La Salle, Ill.: The Open Court Publishing Company, 1918), p. 11.

[2] For the description of a very careful effort to teach geometry from this point of view, see H. P. Fawcett, The Nature of Proof, *Thirteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1938).

[3] J. W. Young, "Fundamental Concepts of Algebra and Geometry" (New York: The Macmillan Company, 1911), pp. 15–25.

equipped to enhance his teaching by a more harmonious and effective coordination of sensible pedagogy and sound logic.

Through its excellence as a vehicle of postulational thought Euclidean geometry lost a great deal of its early historical significance. Herodotus is given credit for the words:

> To teach weak mortals property to scan
> Down came geometry and formed a plan![1]

Geometry was used both in Egypt and Babylonia as an art of measuring in connection with agriculture, irrigation, and architecture; thus it was entirely empirical and intuitive without any idea of logical demonstration. Thales (c. 640–546 B.C.) of Miletus was the first to conceive of the idea of establishing truths through formal demonstration and, in his Ionian school, geometry was not studied for its own sake but as a general preparation for the study of philosophy. It was Pythagoras (c. 572–501 B.C.), a disciple of Thales, and his followers who achieved the arithmetization of geometry in the famous Pythagorean theorem. Plato (429–348 B.C.) introduced to geometricians that powerful tool of the logicians, the analytical method of proof, and one of his most brilliant pupils, Eudoxus (c. 408–355 B.C.), introduced the form of demonstration known as the indirect method of proof.

About 300 B.C. Euclid wrote his famous "Elements" which has served as a foundation for geometry texts during the past two thousand years. In this text Euclid assembled all geometric propositions then existent and added others of his own. His great contribution, however, was the coordination of all known propositions in a simple but logical sequence. This gave to geometry its characterization as a pattern for postulational thinking. Under the influence of Euclid the center of interest in mathematics shifted from Athens to Alexandria. The first Alexandrian school of which Euclid, Archimedes, and Apollonius were members lasted until the beginning of the Christian era, during which time mathematics was studied primarily for the interest in the subject itself. The second Alexandrian school to which belonged such men as Serenus, Menelaus, Ptolemy, Pappus, and Proclus, flourished during the first six centuries of the Christian era. During this period, when utility was the chief objective for the study of geometry, the ideals of education sank to a very low level and an intellectual stagnation set in that continued to prevent any significant mathematical progress until the Renaissance period.

[1] James Gow, "A Short History of Greek Mathematics" (Cambridge, Mass.: University Press, 1884), p. 125.

In order to appreciate the magnificent structure of geometry one not only must be familiar with this period of synthetic development but must realize that there are three other important periods of geometrical history, *viz.*, (1) the period of analytic geometry, foreshadowed by the work of Archimedes (c. 225 B.C.), but in reality exerting a significant influence only after the publication of the researches of Descartes (1596–1650) and Fermat (1601–1665); (2) the period of application of calculus to geometry; and (3) the renaissance of pure geometry which began with the nineteenth century.[1] The teacher of secondary mathematics should have firsthand information concerning these epochs of geometrical history as well as direct contact with the outstanding elements of each. He should know something of the systematized use of algebra that characterizes Cartesian geometry as well as the applications of the techniques of infinitesimal analysis to perfect the study of the relationships existing between the points and lines that generate geometrical configurations. He should know something of the theory of projectivity first enunciated by Poncelet (1788–1867) in 1822 and of Gergonne's (1771–1859) "principle of duality" which so definitely enriches the theory of the united position of point and line. Such names as Ceva (1647–1734), Simson (1687–1786), Euler (1707–1783), Malfatti (1731–1807), Legendre (1752-1833), Monge (1746–1818), Brianchon (1785–1864), von Staudt (1798–1867), and Steiner (1796–1863) should carry significant connotations for him as they represent progressive development in the field of modern synthetic geometry.

Historically, arithmetic developed out of a need for a system of counting, just as geometry found its origin in the necessity for systematic methods of measurement. The teacher of mathematics should know something of the history and nature of number, the foundation of all arithmetic. The beginnings of the concept of number are hidden in the obscurity of time, but there seems to be evidence that some form of number preceded written history by thousands of years.[2] The evolution of our number system has been a real "survival of the fittest" struggle between various scales and systems of notation and is more or less a physiological accident. Had the gods decreed that man should have 6 fingers on each hand and 6 toes on each foot, the human race would probably do all of its counting and calculating on the basis of 12 instead of 10. This would have worked to the advantage of the prac-

[1] D. E. Smith, "History of Mathematics" (Boston: Ginn & Company, 1925), Vol. II, p 331

[2] Tobias Dantzig, "Number: The Language of Science" (New York: The Macmillan Company, 1930), p. 11.

tical man since 12 has more possibilities of factorization than does 10. If, however, the mathematician should decree the base of the number system, he would probably choose a prime, for then there would be economy in symbols, simplification in operations, and eradication of ambiguities.

The definition of abstract number and its elaborate theory possibly forms no part of the necessary preparation of the teacher of secondary mathematics, but he should have a thorough knowledge of the fundamental laws and processes that formulate the theory of integers. The process of counting and the concept of the number of things in a group give rise to the system of positive integers, where, by positive integer, we mean a symbol for the number of things in a group of distinct things. This notion of positive integer creates an abstract symbol for the designation of a characteristic property of groups of objects between which the correspondence is one to one. The fundamental laws of operation and the principle of permanence rationalize the extension of the number system to include negative numbers, fractions, irrational and complex numbers, and transcendental numbers. The historical evolution of this extension with the incidental introduction of the principle of place value, and the concept of a symbol for nothingness, is an epic of enduring significance.

Etymologically, algebra means the theory of restitution (or transposition) and adjustment, in other words a theory of equations.[1] In its modern garb it loses none of this significance but, through its laws of operations upon symbolic forms, expands further into a systematic method for the expression and examination of existing relationships. According to Nesselmann,[2] the history of algebra divides itself into three periods: (1) *the rhetorical*, (2) *the syncopated*, and (3) *the symbolic*. The rhetorical period was characterized by the fact that words were written out in full and no symbols were used. The oldest Egyptian, Babylonian, Arabian, Persian, and Italian algebraists represent this period. In the syncopated period the presentation was similar in literary type to that of the rhetorical period, but abbreviations were used. This period began with Diophantus (c. 275) and extended up to the middle of the seventeenth century. In the symbolic period abbreviations gave way to signs and symbols.

The following examples show this transition from pure rhetorical

[1] Smith, *op. cit.*, pp. 388–390.

[2] G. H. F. Nesselmann, "*Die Algebra der Griechen*" (Berlin: G. Reimer, 1842), pp. 301–305.

algebra to that of modern compact symbolism. They show how the same equation would have been written.

Regiomontanus, A.D. 1464:
3 *Census et* 6 *demptis* 5 *rebus aequatur zero.*
Pacioli, A.D. 1494:
3 *Census* p. 6 *de* 5 *rebus ae* 0.
Vieta, A.D. 1591:
3 *in A quad* − 5 *in A plano* + 6 *aequatur* 0.
Stevinus, A.D. 1585:
3② − 5① + 6⓪ = 0.
Descartes, A.D. 1637:
$3x^2 - 5x + 6 = 0$.[1]

There are, of course, no clear-cut lines of demarcation between the three periods. Diophantus, in fact, used certain features of all three.[2]

The teacher of secondary mathematics should be acquainted with the extended domain of algebra under the freedom of symbolic abstraction. Under such freedom it is possible to define many different algebras[3] and their companion arithmetics. The algebra of the secondary school is the algebra of real and complex numbers. If we restrict this algebra to one of real numbers, we shall have one that is abstractly identical with the metric geometry of three dimensions. This identity is accomplished through the correlation of points on a line with real numbers, which concept was suggested to the mathematical world by Descartes. Among an infinitude of significant achievements this equivalence has enabled modern mathematicians to set up the criterion of constructibility and thus through the algebraic analysis of geometrical relations to prove definitely that it is impossible to trisect an arbitrary angle, to duplicate a cube, or to "square a circle" using *only* the compasses and the unmarked straightedge.

From the secondary-school point of view the graph, equation, and the concept of functional dependence are extremely important. The preparation of the teacher of secondary mathematics should be such that these will be intelligently correlated and understood The part that coefficients of a function play in determining the shape of its graph, the techniques of construction and proper interpretation of the graph and its application to the solution of numerical equations, and

[1] L. Hogben, "Mathematics for the Million" (New York: W. W. Norton & Company, 1937), p. 303.

[2] Smith, *op. cit.*, p 379.

[3] Bell, *op. cit.*, p. 37.

the significance of the graph in the discussion of simultaneous equations constitute some of the contributions that advanced training should make to the professional equipment of the secondary-school teacher. He should know the implications of the fundamental theorem of algebra and some of the more important theorems concerning the relation existing between roots of equations and the coefficients, as well as the application of the elementary theory of determinants and eliminants to the systematic study of equations.

In the words of Prof. E. H. Moore, "functionality is the relation or (mathematical) law of connection between two or more quantities or numbers subject to simultaneous and interdependent continuous variation."[1] This concept of the interdependence of magnitudes was recognized by the early Greeks and Egyptians. Two of the most primitive forms in which we find functional dependence expressed are in area formulas and formulas for the relation between arcs and chords. From these crude beginnings the development was slow, and it was not until the latter part of the seventeenth century that the word "function" was associated with the concept of dependence.

Descartes furnished the crystallizing influence in his discovery and development of coordinate geometry. In 1637 he published his "Geometry"[2] in which he systematized the method of applying algebra to geometry, introduced the notion of variables and constants into the study of geometrical relations, conceived of curves as generated by a moving point, referred these curves to two lines perpendicular to each other, and represented them by equations involving two variables, the relation of these variables being determined by the distances from the two lines of reference. It was this notion of expressing curves by algebraic equations that made possible the step from geometry to analysis and thus paved the way for calculus. A few years later Leibniz introduced the word "function" to designate magnitudes involved in Descartes's idea of curves generated by a moving point. The researches of Newton and Leibniz made extensive use of this new concept of the locus of a moving point and by application of the techniques of infinitesimal analysis gave new significance to the implications of functional dependence.

[1] E. H. Moore, Cross Section Paper as a Mathematical Instrument, *The School Review*, 14 (May, 1906), p. 318.

[2] This work, which appeared as an appendix to the *Discours de la méthode*, was divided into three books. The first dealt with the products of lines; the second defined two types of curves, geometric and mechanic, and treated of tangents and normals; and the third discussed roots and transformations of equations.

Along with the study of analytic geometry the study of trigonometry and differential and integral calculus afford unlimited opportunities for coming into contact with the fundamental principles that are essential to an appreciative and intelligent comprehension of the concept of functional dependence, which, since the early part of the twentieth century, has received increasing emphasis in mathematical instruction. The teacher of secondary mathematics should be familiar not only with the functional analysis of the triangle which trigonometry presents but also with the numerical application and analytic extension of trigonometric techniques in astronomy, surveying, navigation, and calculus. Also he should know how the method of exhaustions, used by Antiphon (c. 430 B.C.), Archimedes (c. 225 B.C.), and others in their efforts to effect the quadrature of the circle, evolved through the method of indivisibles of Cavalieri (1598-1647) and Roberval (1602-1675) into the infinitesimal analysis of Newton (1642-1727) and Leibniz (1646-1716). Finally, familiarity with the power of calculus as a generalizing technique and its importance as a research instrument in pure and applied mathematics should be considered as a significant part of the subject-matter preparation of the teacher of secondary mathematics.[1]

Significant Professional Techniques. The demands made on the teacher of secondary mathematics in the modern program of education make it absolutely essential that he know the orientation of his field of work in the entire secondary program; that he be familiar with significant objectives and problems in secondary mathematics; that he know the techniques of selection of textbooks, workbooks, and other teaching equipment; that he be familiar with the fundamental philosophy of significant evaluation of instruction; that he be skilled in the construction, use, and interpretation of tests—factual and functional, standardized and nonstandardized, objective and essay; that he be acquainted with various instructional techniques and know when and how to use them for maximum efficiency; and that he be prepared to assume his share of the responsibility in the pupil-guidance program.

The teacher of secondary mathematics must be thoroughly familiar with the complementary problems of transfer of training and individual differences. He must know the major sources of student difficulties, how to diagnose these difficulties and plan programs of remedial teaching; he must know how to plan an instructional program and how to adapt this program to different ability groups; he must be enthusias-

[1] For the individual who is interested in pursuing the discussion of this section still further there is provided a list of supplementary readings on pp. 246–247.

tically interested in mathematics and know the fundamental principles of the psychology of motivation; and he should know how to detect and to remedy inefficient study habits and techniques. Finally, the teacher of secondary mathematics should be well versed in the fundamentals of the psychology of learning and in their application to materials and methods for better instruction in mathematics of the secondary school.

We might list the above as the necessary technical equipment of the efficient teacher. As a highly desirable, but not absolutely necessary, part of the program of professional education of prospective teachers we should list skill in the use and interpretation of the techniques of educational research and experimentation. The teachers should be equipped to read, interpret, and evaluate the published results of experimental investigation and to make use of significant findings in the improvement of their own teaching procedures. It is also very desirable that they be equipped to pursue scientific investigations in connection with their own program and to interpret intelligently their findings for the benefit of others.

The Well-prepared Teacher of Mathematics. The foregoing discussion presents what might be called a minimum ideal for the preparation of mathematics teachers. It represents a program which the prospective teacher should at least approximate as nearly as possible. Most beginning teachers, however, must teach in small schools, and in small schools practical considerations generally make it necessary for teachers to teach one or more courses outside their fields of major interest and specialization. Consequently the prospective teacher of mathematics is virtually obliged by circumstances to prepare himself in at least one other field, and this may compel him to forego some of the advanced work in mathematics which would constitute a desirable part of his preparation.

The Joint Commission, in making its detailed recommendations for the training of mathematics teachers, recognizes the limitations thus imposed upon the student's training in mathematics. In view of these limitations it proposes the following as a desirable minimum program:

1. In mathematics:
   a. Courses including complete treatments of college algebra, analytic geometry (including a little solid analytics), and six semester-hours of calculus.
   b. A course that examines somewhat critically Euclidean geometry, and gives brief introductions to projective geometry and non-Euclidean geometry, using synthetic methods (three semester-hours).

c. Advanced algebra, including work in theory of equations, mathematics of finance, and statistics (six semester-hours). This course should give some careful attention to the basic laws of algebra, to the nature of irrational and complex numbers, and operations with them. It should be throughout somewhat critical and not purely manipulative.

d. Either directed reading or a formal course in the history of mathematics and its concepts.

2. In related fields (the related subject is not regarded here as a teaching subject):

An introductory course in physics, astronomy or chemistry that makes some use of mathematics.

3. In professional preparation:

a. A course in methods (two or three semester-hours). This work should be given by a person who has had a good mathematical education and also experience in high school teaching.

b. A course in secondary education (three semester-hours). Some consideration of educational philosophy and of the history of education can be given in this course.

c. A course in psychology, with emphasis on its educational bearing and on the problem of learning (three semester-hours).

d. A course in educational tests and measurements that employs some statistical material (two semester-hours).

e. Practice teaching. It is not usually possible for a student to have practice teaching in two fields. If mathematics is his major he should have practice teaching in that subject.[1]

For those teachers who will teach only mathematics and who will give advanced courses in secondary mathematics, the Joint Commission recommends, in addition to the courses outlined above, as much as possible of the following work:

1. Advanced calculus and differential equations or mechanics (six semester-hours).

2. Additional work in geometry, such as projective geometry, descriptive geometry, etc. (three semester-hours).

3. Additional work in algebra, including some modern algebra (three semester-hours).

4. At least one more of the three sciences of physics, chemistry, and astronomy.[2]

[1] Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, *Fifteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1940), pp. 201–202.

[2] *Ibid.*, pp. 202–203.

For junior-college teachers the Joint Commission recommends the master's degree in mathematics *as a minimum* and also recommends suitable professional preparation.[1]

The program outlined above is substantially in harmony with the program proposed some five years earlier by a Commission of the Mathematical Association of America, Inc.,[2] and is strongly supported by a more recent intensive study of the professional preparation of teachers of secondary mathematics.[3]

The Commission on Post-War Plans states that the teacher of mathematics, whether in the elementary school, high school, or junior college, should have a wide background in the subjects he will be called upon to teach, in related fields, and in the teaching of mathematics. Emphasis is given to the fact that mathematical competence should extend well beyond the level of teaching and that professional understanding should encompass all lower grades. Furthermore, a truly professional program for the training of teachers will take cognizance of the fact that the objectives sought in the training of those who plan to teach mathematics in the elementary and secondary schools are not the same as those sought in the training of research scholars in pure or applied mathematics.[4]

The Co-operative Committee on the Teaching of Science and Mathematics of the American Association for the Advancement of Science made specific recommendations as to the policies of certification of high-school teachers of mathematics and concerning a five-year program for the training of such teachers.[5]

The basic problems of certification of teachers of mathematics have been studied by Layton;[6] those of student teaching by Reeve and Howard.[7] Layton found a wide diversity of practice in the requirements for certification in both subject-matter and professional courses. He was able to make several specific recommendations for more uniform procedures. These recommendations were based on statements

[1] *Ibid.*, p. 203.

[2] The Training of Teachers of Mathematics, *The American Mathematical Monthly*, **42** (1935), 263–277.

[3] Karnes, *op. cit.*, pp. 178–215.

[4] Commission on Post-War Plans, Second Report, *The Mathematics Teacher*, **38** (1945), 215–220.

[5] See this book, p. 41.

[6] W. I. Layton, An Analysis of Certification Requirements for Teachers of Mathematics, *Contribution to Education* 402 (Nashville, Tenn.: George Peabody College for Teachers, 1949).

[7] W. D. Reeve and Homer Howard, Student Teaching in Mathematics, *The Mathematics Teacher*, **40** (1947), 99–132.

from the certification officers of the several states and a large selected group of "specialists in mathematics." Reeve and Howard emphasize student teaching as one of the most important phases in the teacher-training program.

The future of mathematics in the secondary schools is primarily the responsibility of the teacher of mathematics in these schools. Its status will depend largely upon his ability to present and interpret his subject as a worth-while educational venture. The professional preparation of this teacher should equip him with the scholarship and techniques essential to the satisfactory fulfillment of his professional obligation. He must be able to organize and present mathematics in such a way that adolescent boys and girls will be brought not only to a realization of the intrinsic nature and value of mathematics itself but also to an equally clear realization of its role in enabling man to relate, understand, and control his environmental factors and to direct his social and economic advancement.

If these ends are to be attained, the training of the teacher must be an organic and continuous process. It should make the teacher unwilling to permit himself to stagnate under a comfortable self-complacency. It should inspire him to incessant effort both in the expanding of his mathematical horizons and in educational experimentation directed toward the improvement of his instructional techniques. The teacher so prepared will not restrict his attention and his instruction to the confines of mere operational mechanics. He will lead his students with contagious enthusiasm into realms of mathematical thought and endeavor which will be both stimulating to their curiosity and intellectual interest and broadly significant to their insights, appreciations, and general cultural development.

Exercises

1. Contrast the implications of the following two statements: (*a*) Mathematics is a tool subject. (*b*) Mathematics is a fundamental mode of thinking. Give three illustrations of each of these concepts of mathematics.

2. In the teaching of secondary mathematics what emphasis should be placed on mathematics as a tool subject? What emphasis should be placed on mathematics as a fundamental mode of thinking?

3. What have been some of the major influences that have brought about the employment of so many poorly prepared teachers?

4. What range of requirements will meet the demands for state certification to teach secondary mathematics? (*Cf.* footnote 6 on page 242.)

5. Why should the teacher of secondary mathematics be thoroughly familiar with the mathematics programs of all lower grades?

6. What contribution can the history of mathematics make to the better preparation of teachers of secondary mathematics?

CHAPTER X

SUPERVISION OF INSTRUCTION

In nearly every enterprise which engages the services of any considerable number of individuals, the supervision of the work of these individuals is regarded as an important function essential to the economical and efficient operation of the enterprise. Engineering and construction crews have their foremen, large stores their departmental managers, restaurants their headwaiters, large offices their chief clerks, governmental bureaus and departments their supervisors, and so on through an endless list. It may be taken for granted that those in charge of the operation of business enterprises would not engage the services of these individuals unless they felt that the returns justified the expense.

Viewed from a strictly financial standpoint the business of public education is the largest single business in this country. While it must be regarded as more than a mere business enterprise, it does entail the expenditure of vast sums of money for the primary purpose of instruction. It seems reasonable, therefore, to insist that this instruction be made to yield as large a return as possible. There is both theoretical and experimental justification for the belief that direct supervision can contribute much to the improvement of instruction in the secondary school and thus increase the efficiency of the entire educational program.

The supervision of secondary-school subjects has certain broad functions and employs certain broad principles and types of activities which are pretty much the same whether applied to the teaching of mathematics, art, history, or any other subject. It is difficult, if not impossible, to give any adequate discussion of supervision apart from these. The purpose of the present chapter is to consider these basic functions and principles as they apply to the supervision of mathematics. Generality cannot be avoided in the discussion, but, if the reader will maintain the point of view of the individual responsible for the functioning of a *mathematics department*, the application of the discussion to this particular responsibility will be apparent throughout the chapter.

The Need for Supervision in Secondary Mathematics. There is need for the supervision of the work of mathematics teachers in the secondary school. Large numbers of these teachers come into the schools each year fresh from college, with high enthusiasm for their work but with little or no experience in teaching and without much firsthand knowledge of the problems connected with teaching secondary-school students. These problems cannot be adequately mastered in the course of one or two years, even under favorable circumstances. Most young teachers can be greatly helped by the guiding advice and counsel of a sympathetic and experienced supervisor for a period of several years after they enter upon their work, and, since the annual turnover in the teaching staffs of secondary schools is relatively high, this period may in many cases include the entire professional career of the teacher.

The need for supervision is accentuated by the fact that the majority of secondary-school teachers find it necessary to teach one or more subjects in fields other than those of their major preparation. Thus many mathematics classes are taught by individuals whose primary interest and preparation have been in other fields and whose training in mathematics has been largely of an incidental nature. Complete familiarity with the subject matter to be taught, while not a sufficient guarantee of teaching proficiency, is certainly a necessary condition, and any limitation of such familiarity evidently imposes a severe additional handicap upon the teacher.

Extensive and reasonably successful experience is a great asset to the teacher of mathematics, but such experience itself is sometimes not without its potential disadvantages. In particular, the teacher who has been successful in his work may find it easy to get into a rut and to become complacent about his work and oblivious or insensitive to possibilities of further improvement. Thus even teachers of long experience and substantial training may benefit from appropriate supervisory contacts through being stimulated to professional alertness and scholarly advancement.

In spite of the foregoing considerations, however, if one may judge from its prevailing status, no great amount of importance has generally been attached to supervision of mathematical instruction in the secondary schools. Even its functions and aims have not always been properly understood. As a result, supervision, where it exists at all, is all too often carried on in a sporadic and more or less planless fashion which accomplishes little of value. There are certain situations, of course, more often encountered in the secondary schools of large cities,

where effective supervision exists. On the other hand, in many of the larger schools the supervision is often perfunctory and fails to achieve the major outcomes which are properly implied in the term, while in a great many smaller schools supervision is practically nonexistent.

There are several reasons why this is true. Often such supervision as is carried on must be done by the superintendents, principals, or department heads who have many duties other than those relating to supervision. Department heads must be concerned largely with their own classes and with administrative affairs of the department, while the professional training of the superintendent or the principal of a school will usually have been concerned more with administrative matters than with principles and techniques of supervision. It may be said also that the postponement or neglect of administrative and instructional duties is much more likely to result in immediately and obviously embarrassing situations than is the neglect of supervision. Consequently, in view of universally crowded time schedules and in view of the usually inadequate training in the principles and techniques of supervision, there frequently exists the tendency to subordinate supervisory activities to other matters which may appear to be more immediately pressing. Moreover, teachers often dislike the idea of supervision. Failing to appreciate its primary function, which is the improvement of instruction, many teachers regard supervision merely as personal criticism and as a means through which ratings and possible invidious comparisons are to be made. Where this point of view persists, teachers are likely to oppose any attempts at supervision and to look upon their supervisors with suspicion and distrust. Such an attitude nullifies the beneficial effects toward the attainment of which all supervisory effort should be directed.

If supervision is to result in improved instruction, it must be a cooperative effort. Teachers and their supervisors must have a mutual and sympathetic understanding of the reasons for supervision, of its aims and functions, and of the advantages which may be expected to result from it. Such a feeling provides the only sound basis for thorough cooperation in the enterprise, and it is only through such cooperation that lasting and helpful results may be expected.

Functions of Departmental Supervision. Any means through which the mathematical instruction in a school may be improved is a legitimate and proper function of departmental supervision. It may be taken as axiomatic that the training and competence of the teacher, together with his attitude toward his work and his students, are more influential than any other considerations in determining the effective-

ness of instruction. Therefore the most obvious and direct, and perhaps the most important, service which departmental supervision can perform lies in giving that direct counsel, guidance, stimulation, and assistance which will encourage the various teachers in the department to continue their professional development through the study of pertinent subject matter and professional literature. This aspect or function of supervision may be designated as the in-service training of teachers.

There are, however, other ways in which the proper supervision of the department may contribute appreciably, if less directly, to the improvement of instruction. These consist mainly of the provision and maintenance of adequate instructional facilities, the maintenance of cordial and cooperative professional relations with the administrative offices of the school, and the prosecution of research and the dissemination of professional information bearing upon the instructional problems of the department. As illustrative of these other functions of supervision may be mentioned such matters as the following:

1. Selecting and organizing suitable teaching materials for the several classes
2. Preparing courses of study and coordinating the work of the department
3. Comparing the suitability of different textbooks, workbooks, and other published materials for use in instruction
4. Comparing the efficacy of different methods of instruction
5. Establishing and defining suitable standards of attainment for the various courses in the department
6. Planning, inaugurating, and carrying on a systematic program of evaluation of student attainment
7. Maintaining a suitable, convenient, and facile system of departmental records
8. Holding departmental meetings for the consideration of matters of common concern
9. Keeping the members of the department informed with reference to the objectives and activities of the supervisory program
10. Conducting research designed to secure data which may provide sound bases for the improvement of materials or methods of instruction
11. Locating and abstracting worth-while pertinent articles, research studies, etc., and making the substance of these available in condensed and understandable form to the members of the departmental staff
12. Cooperating with those responsible for pupil guidance
13. Keeping the administrative officers of the school informed with reference to the work of the department and maintaining with them reciprocally cooperative relations

14. Keeping the administrative officers of the school informed with reference to departmental needs and making suitable recommendations concerning provision for those needs

15. Evaluating the effectiveness of instruction and submitting the results of such evaluation, together with appropriate recommendations, to the administrative officers of the school

The functions of departmental supervision have been variously classified by different writers.[1] Some of them are mainly of an administrative nature, while others emphasize the coordination of interdepartmental effort, the conducting and reviewing of research work, the organization of instructional materials, the rating of teaching efficiency, the consideration of special problems, and the in-service training of teachers. They all bear either directly or indirectly upon the general problem of improving the quality of instruction.

Improvement of Instruction in Mathematics through Proper Organization and Articulation of Courses. Much of the criticism which has been directed at mathematical instruction in the secondary schools has emphasized the view that the courses are separated into "watertight compartments" and that the program as a whole has lacked unity and continuity. This charge has been made especially with reference to the courses in algebra and geometry. Efforts have been made to correct this situation by organizing courses in general mathematics. These efforts have been notably successful in the junior-high-school courses, especially those for the seventh and eighth grades. In fact, general mathematics has come to be the typical offering for these grades. In the ninth grade, algebra still enrolls more students than general mathematics does, though there are indications that more than half of all ninth-grade students may choose between algebra and general mathematics.[2] Efforts to organize the courses of the senior high school into a thoroughly integrated sequence have not been notably successful, but their interplay is being emphasized more and more in the separate courses. Considerable progress has been made toward integrating the junior-college courses, and efforts in this direction continue to gain momentum.

The importance of continuity in mathematical instruction cannot be too greatly emphasized. It is unfortunate for students to feel that

[1] E. R. Breslich, "The Administration of Mathematics in Secondary Schools" (Chicago: University of Chicago Press, 1933), p. 5; also C. W. Knudsen, "Evaluation and Improvement of Teaching" (New York: Doubleday & Company, Inc., 1932), p. 2.

[2] Raleigh Schorling, What's Going On in Your School? *The Mathematics Teacher*, 41 (1948), 147–153.

the mathematics of the junior high school is unrelated to that of the senior high school and the junior college, or to feel that algebra, demonstrative geometry, and subsequent courses have no points of contact with each other or with the earlier work in arithmetic, intuitive geometry, and the informal algebra of the seventh and eighth grades. This feeling, however, is all too prevalent, not only among students but among teachers as well. Most teachers of secondary mathematics have received their training in traditional segregated courses and have not been made emphatically aware of the interrelations of the different branches of mathematics. As a result they, in turn, often fail to emphasize these interrelations in their teaching.

The supervisor may do much to repair this defect in the teacher's point of view. He should point out many instances illustrating the bearing of one course upon another. In particular he should give convincing illustrations of the fact that the more specialized courses which are usually offered in the ninth grade and beyond have their foundations in the earlier courses of the junior high school and that, conversely, many of the mathematical concepts and principles developed in these earlier courses find some of their most important and interesting applications in the work of subsequent specialized courses. It should be made apparent to the teacher that the full enrichment of the work at *any* level of the secondary school cannot be attained unless the teacher has this perspective and is able to envisage the mathematical program in its entirety, with its manifold interrelations. The supervisor who has been able to bring all his teachers to this point of view will have done a great deal to improve instruction all along the line.

From what has been said, it is obvious that the supervisor and the teachers in the mathematics department should be given a good deal of freedom in determining what subject matter should be included in the various courses, and in organizing the courses of study for the different grades. Unless they are given such freedom, it may not be possible to realize fully the potential benefits which have just been described. Supervision must have a hand in curriculum construction if it is to justify itself completely. To shackle it in this respect is to impair its usefulness.

Improvement of Instruction through In-service Training of Teachers. Of all the functions and activities of the teacher of mathematics that of conducting the regular classroom instruction bears most directly upon the degree of student mastery of subject matter and its applications and upon their attitudes toward the study of mathematics. Since these represent the ultimate objectives of all mathematical instruction,

it follows that the most immediate and important function of supervision is the examination and improvement of those activities and characteristics of the teacher which are involved in actually conducting the work of the class. In broad outline these include the following:

Planning units of instruction and daily class period activities
Selecting and organizing the materials of instruction
Direct expository teaching
Making assignments
Directing study and training students in effective methods of study
Preparing and supervising appropriate drill, review, and maintenance work
Appraising the achievement of students
Diagnosing difficulties and applying appropriate remedial procedures
Adapting instruction to individual differences
Providing motivation of effort
Handling routine matters and details of class management in an economical and effective manner

There are also other activities and considerations which, although they are perhaps only indirectly related to the work of the classroom, have definite bearing upon the effectiveness of this work and upon the success of the teacher. Among these may be mentioned the comparison and evaluation of textbooks, workbooks, and other instructional materials; the careful study of individual aptitudes with a view to making a contribution to the educational guidance of the individual students; earnest cooperation with the administrative officers of the school; continuance of interest in professional and academic self-improvement; and the development and maintenance of those personal characteristics which command the respect and cooperative good will of the students.

The need for the continual training of teachers in service is evident. Even if it might be assumed that all teachers are sufficiently familiar with the subject matter of their courses, many of them lack the experience necessary to give them poise and self-assurance and to produce that familiarity with the various techniques and devices which they must have to enable them to plan and conduct the work of their classes with discriminating judgment. Others, more mature in point of experience, will have developed more or less fixed patterns of thought and work which with the passage of time tend to become less flexible and adaptable to circumstances.

Some tend to become radically enthusiastic over each new technique, device, or point of view, regardless of whether or not its validity and its practicability have been established and confirmed. Such teachers need to be trained to develop a more penetrating and critical viewpoint

which will be characterized by stability and conservatism as well as by open-mindedness and receptiveness to new ideas. They need to gain the realization that there is no single new panacea which will solve all the problems of teaching. Others, by way of contrast, will have become so conservative in their viewpoints as to be unwilling to admit any virtue in anything that is new, or even to try out in an experimental way suggestions for the improvement of their work. They need to have their attitudes modified and liberalized, to become receptive to new ideas while remaining properly critical of them, and to realize that improvement as well as conservation is desirable.

Some teachers are too easygoing, and do not maintain proper order in their classes. In such circumstances it is impossible for either the teacher or the students to work to best advantage. Interruptions occur, time is wasted, attention is dissipated, and little is accomplished. Students are not slow to size up situations of this kind, and, where they occur, the teacher is likely to forfeit a large measure of the respect of his students. This, of course, is about as disastrous a thing as could occur. On the other hand, some teachers tend to be "hard-boiled" and to lack sympathetic appreciation of the motives and reactions of the students. They need to be helped to a realization that the teacher must be more than a disciplinary officer, that freedom and discipline are not inconsistent, and that the highest type of orderliness is not a product of inhibition but the outgrowth of a cooperative attitude. They need to learn that more can be accomplished by wisely guiding the energies of the students into productive channels than by merely suppressing them.

The foregoing are but examples of situations which might be improved through the in-service training of teachers under the counsel of wise and sympathetic supervision. Many others could be given: the planning of work, the construction of tests, the keeping of records, various details of class management, the clarification of the teacher's relations and obligations to the administrative department of the school and to the community, the general relating and adjusting of the teacher's personality and educational philosophy to the immediate instructional situation, and in particular the class-period activities of the teacher. All these present problems upon which teachers need suggestions from time to time; most teachers welcome these suggestions if they are given in a tactful manner and helpful spirit. There is no way in which supervision can more materially and directly contribute to the improvement of instruction or can more amply justify itself than through this in-service training of teachers.

This phase of supervision must be carried on largely through class visitation and conference and at times through demonstration lessons taught by the supervisor. Before the supervisor visits classes, he should make it clear to the teachers that his purpose in visiting is not primarily to criticize, restrict, or prescribe their work, but to help them in every way possible to improve their teaching and to find greater satisfaction in it. Because supervisors are not always careful to do this, teachers often have a dread of the supervisor's visits, and, because of the inhibitions and tensions that are thus set up, they fail to do themselves justice. The supervisor can do much to avoid this unfortunate situation if he will preface his visits by laying a groundwork of mutual understanding and confidence.

The supervisor should make his visits as informal as possible, but they should not be haphazard. He should strive to avoid making himself conspicuous, and the class should learn to accept his presence in the room as a perfectly normal thing calling for no detraction of their attention. It is perhaps well to let the teacher know in advance of some of the visits, and especially is this true with reference to the first few visits to inexperienced or new teachers. However, after a time the supervisor should feel that he can come into the classroom at any time without producing any feeling of tension or restraint in the teacher or the students. Above all, the supervisor should avoid doing anything to embarrass the teacher in the presence of the class. It is nearly always a mistake for the supervisor to volunteer suggestions to the teacher during the class period. Any suggestions which he has to make should be mentally noted, but the time to bring them up for discussion is in the subsequent conference and not during the class period. Nothing can destroy the cooperative attitude of the teacher more quickly or more surely than the practice of placing him at a disadvantage in the presence of his students.

As a rule, visits to classes should be followed by conferences with the teachers. These not only give opportunity for the discussion of points where improvement seems desirable and for suggestions toward such improvement, but they also offer to the teachers opportunities for raising questions and for seeking advice. The conference is the main avenue to a wholesome and friendly mutual understanding between supervisor and teacher, as well as to specific improvements in the instruction. Without conferences no constructive benefit can come from supervisory visits. Indeed, the supervisor who visits classes but does not confer with the teachers afterward may actually engender thereby a feeling of active antagonism, since in these circumstances

the teachers, not unreasonably, may conclude that the supervisor is merely sitting as a judge upon them and not as a helper to whom they may look for counsel and guidance in their work.

The length of the conference will depend upon the nature and the seriousness of the matters to be discussed. Conferences should be scheduled, as far as possible, to suit the convenience of the teacher rather than that of the supervisor. During the conference it is quite as important for the supervisor to comment upon the favorable impressions which he has received as it is for him to offer criticisms and suggestions on points which need improvement. The supervisor should avoid being arbitrary. He should make every effort to justify his views to the teacher, and he should give equal opportunity for the teacher to justify the teaching procedures used or to make other suggestions even if these should run counter to his own ideas. Conferences held in this spirit of freedom and open-mindedness can render the program of visitation extremely helpful and stimulating both to the teacher and the supervisor, thus giving to the program the vitality and the truly functioning force which characterize supervision at its best.

Improving Instruction through a Program of Testing and Associated Activities. Testing has been in use as long as education has existed, but it has been used in the schools primarily as a means for assigning marks. Only in recent years has there come a general recognition that the use of tests may serve other important ends. It is now universally recognized, however, that tests can be used not only to measure the results of instruction but also to *improve* instruction, and there is a widespread and growing feeling that this, after all, should be regarded as the most important service which they can render.

In Chap. VIII there has been given a discussion of the philosophy underlying the testing program in connection with schoolwork and of the principles of selection, construction, use and interpretation of tests. Consideration has been given to the functions, nature, and techniques of prognostic testing as a means to effective guidance, to the use of inventory and achievement tests as measures of accomplishment, to the use of practice or drill tests as devices for the organization and fixation of details and processes and for the perfection of skills, and to the role of diagnostic testing in indicating and facilitating needed remedial work.

The important consideration here is that all these particularized forms and uses of tests are fundamentally pointed toward the improvement of instruction. Since this, in turn, is the primary function of supervision, it must follow that the planning of a comprehensive test-

ing program and the advising and guiding of the teachers in the administration of such a program is an important part or phase of supervisory work.

Many teachers have not had the technical training and the experience which are necessary for mapping out a comprehensive testing program wisely, for administering it systematically, and for translating the findings and the implications into appropriate instructional procedures. It is an important function of supervision to familiarize the teachers with the ways in which they can make tests contribute to the improvement of their teaching, and to offer them constructive, practical suggestions with regard to the selection or construction of appropriate tests and to the interpretation and constructive use of the results. This should be, in fact, an important part of the in-service training of teachers, and as such it must be regarded as a definite responsibility of the departmental supervisor.[1]

Improvement of Instruction through the Encouragement of Teachers to Seek Self-improvement. One of the major contributions which supervision can make to the general improvement of instruction is the stimulation of a desire for self-improvement on the part of the teachers along both academic and professional lines. That many teachers in the secondary school are inadequately prepared for their work has been noted by countless observers. In spite of rising standards and increasingly rigorous requirements for certification and assignment, the fact unfortunately remains that many teachers must teach courses which are not in their own fields of specialization and for which, in many cases, they have little or no academic background. Moreover, the heavy teaching loads which prevail almost universally in the secondary schools and the multitude of extracurricular duties and activities which teachers are expected to assume exact so much time and effort that teachers have little opportunity for deliberation and reflection. Continual attention to the urgent demands of the moment tends to become a habit which may easily obscure the longer view of the job, with the result that, even when opportunities for systematic study with a view to academic and professional improvement do occur, they are likely to be overlooked or dissipated through the habitual focusing of attention upon immediate details rather than upon larger issues.

This is unfortunate, because instruction cannot reach its maximum effectiveness unless those who give it continue to expand the breadth of their scholarship in the subject-matter fields where their teaching

[1] Paul B. Jacobson, The Place of Testing in the Supervisory Program, *University of Illinois Bulletin*, **35**, No. 89 (1938), pp. 1–34.

lies and to give conscious and sustained attention to the improvement of their techniques of teaching. The best teaching is that which is intellectually stimulating to both the students and the teacher, and really stimulating teaching is seldom done by a teacher whose perspective embraces no more than the immediate details of subject matter which he is to teach and the routine activities of the day's work.

The fact that a great many teachers do work under these limitations is not necessarily their own fault. Many of them lack the maturity and experience which would enable them to organize their work more efficiently so that they might find time for systematic advanced study and professional reading, while others lack the vision which would stimulate them to want these things.

Opportunities for self-improvement are not lacking, but they are often obscured by the conditions which have been mentioned. One avenue to academic and professional advancement is attendance at the summer sessions which are held in universities and teachers colleges all over the country. Great numbers of teachers do take advantage of this opportunity every year. Summer schools, however, do not by any means represent the only avenue to self-improvement. Many universities offer both academic and professional work through the medium of correspondence courses and sometimes through extension classes. These courses are relatively inexpensive, and they serve to systematize study and to give it organization and continuity. The teacher who for any reason finds it impossible to pursue any of these regularly organized courses can still plan for himself a systematic program of study or professional reading or can undertake an organized study of certain aspects of his own work. Definite benefit may be derived from such activities as these, and they will do much to make his teaching more stimulating and beneficial both to his students and to himself.

The unfortunate thing is that most teachers become so engrossed in their immediate problems that the possibilities of carrying on any systematic program of self-improvement escape them. For this reason it should be one of the major objectives of the supervisor to give to his teachers suggestions and continual encouragement in this direction. He should help the young and inexperienced teachers to plan and organize their work so that it may be carried on with as much economy of time as may be consistent with the requirements of the situation. He should keep the experienced teachers aware of the possibilities of improving their techniques and of enriching the content of their courses. He should keep all the teachers conscious of the importance

of both professional and academic growth and should do everything possible to encourage and facilitate their efforts to bring themselves to an ever higher level of competence.

The Place of Research in Supervision. Research has a dual role to play in supervision. Some of the practical instructional problems which face the teacher may find their solutions or partial solutions in the published results of research that has been carried on elsewhere. On the other hand, such problems may legitimately give rise to research in connection with the work within the department itself. In the one case supervision goes to outside sources for help in answering its questions; in the other, it endeavors to answer them for itself.

There has been much experimental work done in the field of the teaching of mathematics. The published reports of these investigations, however, are for two reasons often of little direct benefit to teachers in service. In the first place, many of them are not readily accessible except in the libraries of colleges and universities. Secondly, many of them are so replete with technicalities and detail that most teachers have neither the time nor the technical background needed to evaluate the validity of the procedures employed and to isolate and interpret the significant findings so that they can be translated into practice. Teachers in active service need to have the results from such studies made available in concise, understandable, and usable form. Therefore it is a function of the departmental supervisor to familiarize himself with the published researches which are really significant, to digest them, and to abstract and summarize the important findings of these investigations, making them available to the teachers with suggestions for translating them into practice and incorporating them in their instructional procedures.

Many instructional problems, however, can be investigated by the teachers themselves, and such investigations often turn out to be not only useful in themselves but extremely stimulating to the teachers who conduct them. As a rule they will necessarily be limited in scope, and the findings may be of local, rather than general, importance. In some quarters they might not even be dignified by the title of research. This, however, is beside the point. The important thing is that they represent efforts on the part of the teachers to apply scientific procedures to the solution of instructional problems, and in this way they serve to make the teachers acutely aware of the problems and to lead them to make careful analyses pointing toward practical solutions. The following list represents a number of random suggestions of such problems. It could be indefinitely expanded, since every teacher who

trains himself to be "problem conscious" will find many other instructional problems arising in connection with his own work. Perhaps, however, these suggestions may help teachers to become more sensitive to the presence of such problems and to the possibilities for investigating them systematically.

Compilation of a list of specific uses of mathematics in everyday life situations
Compilation of a list of specific uses of mathematics in industry
Compilation of a list of specific uses of mathematics in other fields of work or study
Study of the effect of various devices for the motivation of junior-high-school mathematics
Analysis of errors in algebra to determine causes
Comparison of treatment of certain topics in algebra in different textbooks
Comparison of sequences of theorems in different textbooks in geometry
Experimentation with variations of method in teaching verbal problems
Pretesting and growth studies in solid geometry
Study of the effect of the systematic preparation and use of diagnostic tests in plane geometry
Devising appropriate activities in geometry for superior students
Comparison of results of supervised study and the recitation method in first-year algebra

Sometimes the efforts of several teachers or of the entire departmental staff may be enlisted in cooperative studies of this sort. Such an arrangement tends to give added interest to the work and to increase the reliability of the results. It is quite possible for investigations conducted by the teachers themselves to yield objective results that will be practically helpful to them in their own work. However, an even greater benefit is derived from the stimulation of professional interest which almost invariably comes about as a result of participation in such investigations.

It is a function of supervision to envisage significant instructional problems, to propose them to the teachers for study, and to cooperate with the teachers in the formulation of methods for their investigation. The departmental supervisor should take the lead in this. He should have the most comprehensive view of the problems which should be studied and the most discriminating judgment as to which ones are suitable for investigation by the teachers. The administrative officers of the school are not likely to be sufficiently informed about instructional matters within the department, while the teachers are likely to be too greatly preoccupied with the details of their immediate duties to give much attention to the matter of initiating research. The departmental supervisor alone stands in an altogether favorable position to

sense fully the need for these activities, to enlist the cooperation of the staff members and, if need be, of the administrative officers, and to give the greatest measure of encouragement, guidance, and assistance to the teachers in the prosecution of the investigations.[1]

Evaluation of Instruction and the Rating of Teachers. It has been emphasized throughout this chapter that the fundamental purpose of supervision is the improvement of instruction. Obviously, there can be no basis for constructive suggestions for improvement except through the appraisal of the instructional practices and the qualifications and characteristics of the individual teachers. Therefore the evaluation of instruction and the rating of teachers are necessary functions of supervision, and the responsibility for carrying on the activities incident to these functions must devolve mainly upon the departmental supervisor.

Evaluation and rating will be meaningful and helpful only to the extent to which it succeeds in indicating points of strength and points where improvement is needed. It must be, in fact, a sort of diagnostic process applied to the traits and teaching activities of the teacher. Rating charts of one form or another are commonly used to facilitate and objectify the procedure. They usually consist of lists of items or characteristics which are believed to be important in relation to success in teaching. The majority of the items in most of these rating lists refer to characteristics or traits which are important elements in connection with the teaching of any subject. In a study of 57 of these rating devices Knudsen[2] has noted 79 of these traits, and of these, not a single one applies uniquely or with special force to teachers of any particular subject. Occasionally, however, there may be found rating devices which have been constructed specifically for the purpose of analyzing teaching in a particular field. Such lists are more definitely helpful to the departmental supervisor in appraising the work of the teachers in his own department than are the lists of the more general traits.

It is important that teachers should have a correct point of view with regard to teacher rating. Too often they feel that its only purpose is to detect incompetencies and that the detection of these incompetencies will result only in censure, demotion, or dismissal. It is true that ratings may be used legitimately for administrative purposes involving matters of tenure, status, and salary adjustments, but formal objective

[1] Paul V. Sangren, The Participation of the Classroom Teacher in Educational Research, *Educational Administration and Supervision*, 15 (1929), 593–601.

[2] Knudsen, *op. cit.*, pp. 212–215.

ratings, when employed for these purposes, should be used only in connection with other considerations. They should never be made the sole bases for these administrative adjustments, and this should be made clear to the teachers. The only way to secure a cooperative and helpful attitude toward teacher ratings is to make sure that no misunderstandings exist as to the main purpose of these ratings, *viz.*, the improvement of instruction. Such misunderstandings can best be avoided by encouraging the teachers themselves to participate in the rating process. If rating lists are to be used, they should be made available to the teachers, and the teachers should be encouraged to use the lists in analyzing their own work. In this way they may locate and identify deficiencies in their own teaching. The supervisor should also make ratings of the teachers and should frequently discuss the ratings with the teachers individually. These discussions, if carried on sympathetically and tactfully by the supervisor, will go far to dispel any feeling of antagonism toward teacher rating and to bring about improvement with respect to those traits and activities which are found to need special attention

Making Supervision Effective. The success of any program of supervision will depend eventually upon the characteristics and the activities of the supervisor. It has been emphasized that, if supervision is to eventuate in any thoroughgoing improvement of instruction, it must be a cooperative enterprise in which the cordial participation of the teacher is an indispensable element. It follows that the first major objective of the supervisor must be to gain the confidence of the teachers and to acquaint them with the real purposes of the program. This means in effect that the teachers should be made partners in the enterprise and that they should enjoy the full confidence of the supervisor. This will beget mutual understanding and freedom in the interchange of ideas and suggestions and will tend to prevent or dissipate the distrust and suspicion which sometimes unfortunately mark the attitude of teachers toward supervisory activities.

In order to develop this cooperative attitude successfully, the supervisor must be a person of culture, insight, academic and professional ability, and vision. He should have had extensive experience in teaching mathematics in order that he may know and appreciate the many problems which the teachers actually must face in their work and that he may appraise with sympathetic understanding their methods of handling these problems. He must be tactful in his efforts to assist them. He should have a breadth of knowledge and a degree of mastery of the branches and applications of mathematics which will

command the respect of the teachers and out of which he may help them to broaden and enrich their academic perspective. He must be familiar with educational movements and developments, both in general and in the field of mathematics in particular. He must know how to appraise educational movements, and he should make their implications clear to his staff.

He should be a master teacher, because there will be times when it will be desirable for him to teach demonstration lessons. The teachers in the department almost certainly will look with skepticism upon the suggestions of one who is unable to incorporate in his own practice those things which he suggests to others. On the other hand, he should manifest at all times a willingness to learn from the teachers and should acknowledge readily his appreciation of any helpful suggestions which they are able to give him.

Finally, he must have organizing and executive ability, initiative, and professional vision, because it is he who must take the lead in initiating and guiding the activities directed toward the improvement of instruction within the department and in coordinating and harmonizing the work of the department with the whole educational program of the school. The supervisor who is equipped with these qualifications should have little difficulty in building up a wholesome atmosphere of interest, respect, cooperation, and good will toward the supervisory program, both among the members of the department and among the administrative officers of the school.

A Program for Self-supervision. While this chapter has outlined the functions and methods of supervision in mathematics primarily from the standpoint of the departmental supervisor in a large school, the fact has not been overlooked that most secondary schools are not large schools and do not have special departmental supervisors. The problems of supervision, however, are the problems of the improvement of instruction, and these exist in all schools, large or small. Therefore the intention has been to discuss these problems comprehensively but simply and clearly, so that the suggestions which have been made will not necessarily require the services of a special departmental supervisor to translate them into practice but can be interpreted and adapted to local situations by any member of the school's staff.

In most cases whatever special supervision is given at all must be given by the superintendent or the principal of the school. It is believed that the suggestions contained in the foregoing pages will be helpful to these officers in carrying out this phase of their work. There

are many cases, however, where even the administrative officers are so burdened with teaching and with inescapable administrative duties that they literally can find no time for supervision. In such cases the responsibility for the improvement of instruction must be assumed by the teachers themselves.

In the matter of self-supervision the teacher's attention should be confined mainly to the improvement of his instructional activities, to the extension of his command of subject matter, and to the cultivation of a professional attitude. There is much that he can do in these directions if he is willing to appraise his own work honestly and to try to improve the parts of it which appear to him to be most in need of improvement. First, he can analyze his personal traits, his teaching conditions, and his teaching activities. He can make a rating chart by means of which he can subject himself and his work to detailed scrutiny and can give it an honest and careful rating. A list of items of the type included in Breslich's suggestions for supervisors' visitation records[1] would form a simple and practical basis for such an analysis and rating.

Having rated his work in this way, he should concentrate his attention first on a few of the important points which seem to be most in need of improvement. These should be analyzed further, if necessary, in an effort to determine in what particulars and in what ways they can be improved to best advantage. After this is done, the teacher will be in a position to make a sound beginning at actually bringing about the desired improvements. It is better not to work on too many points at once. Each fault which is to be corrected will require special attention for a period of time, and the attempt to focus special attention on too many things at a time might result in a dissipation of effort, which, in turn, might be actually detrimental rather than beneficial to his teaching. On the other hand, the teacher will find that well-planned, systematic, and conscientious effort directed toward the correction of a few teaching practices at a time will have a cumulatively beneficial effect and will in the end far more than justify itself.

Finally, in the scheme of self-supervision the teacher can set for himself a systematic program of professional or academic study. Suggestions along this line will be found in an earlier section of this chapter. Here, again, not too much should be attempted at once, but such a program, if moderately planned and consistently pursued, will give added interest and enrichment to his work and will bring an increased measure of satisfaction.

[1] Breslich, *op. cit.*, pp. 15–16.

Exercises

1. Contrast the need for supervision of mathematical instruction in American secondary schools with the present status of such supervision.

2. Explain why teachers often look upon supervisory activities with suspicion and dislike, and discuss measures which should be taken to counteract this attitude.

3. Enumerate the characteristics or traits which the supervisor should have in order that he may carry on his supervisory program to best advantage.

4. Enumerate matters which would be especially suitable for discussion at the first departmental meeting of the school year.

5. Make another list of topics which could profitably be considered at subsequent departmental meetings.

6. Explain how the proper organization and articulation of the courses in mathematics can contribute to the effectiveness of instruction.

7. Why should both the teachers and the supervisor have a hand in planning the courses of study and selecting textbooks and equipment?

8. Describe explicitly the ways in which the supervisor may help the teacher in planning his work for the year.

9. Discuss fully the reasons why the in-service training of teachers should be regarded as perhaps the most important function of supervision

10. Adapt or construct a checking list for rating the instructional procedures of teachers of junior-high-school mathematics.

11. In what respects, if any, would you alter this checking list to make it particularly adaptable to rating teaching procedures in senior-high-school or junior-college mathematics. Justify your suggestions for alteration, or your decision to make no changes in the list.

12. Describe clearly the ways in which prognostic testing may contribute to the improvement of instruction in mathematics. What are the limitations and potential dangers in the use of prognostic tests?

13. Give a detailed discussion of the ways in which diagnostic testing can contribute to the improvement of instruction in mathematics. Illustrate if you wish.

14. Discuss the ways in which research may contribute toward making supervision effective. What are the supervisor's responsibilities with reference to research in the supervisory program?

15. Summarize one comprehensive research study which has bearings on the teaching of secondary mathematics, and point out the instructional implications of the study.

16. Select one instructional problem which might be investigated by a departmental staff, state its implications, and outline the procedure which you would recommend for conducting the study.

17. Discuss the responsibilities of the supervisor and of the teachers in relation to the educational guidance of students.

18. Discuss the supervisor's responsibilities in the matter of encouraging teachers to seek academic and professional self-improvement, and point out possible avenues to the achievement of this aim. What are the chief obstacles in the way of such self-improvement?

19. Enumerate and discuss the ways in which a complete and convenient system of departmental records may contribute to the effectiveness of supervision.

20. Draft a set of forms for the records which you would want included in such a system if you were the supervisor.

21. Explain clearly the particular ways in which the supervisor can contribute to the improvement of instruction through his status as a liaison officer between the department and the administrative officers of the school.

22. Explain why it is an important function of supervision to see that adequate facilities and favorable working conditions are provided for the teachers.

23. If you were the supervisor of a mathematics department and were required to submit ratings of your teachers to the administrative officers, upon what items would you feel it proper to rate them? Construct a rating scale including these items.

24. What specific advantages could result from the use of such a rating scale? What dangers might it involve? What measures could be taken to minimize these dangers?

25. Why should formal ratings of teachers not be used as the sole basis for promotion, retention, demotion, salary adjustments, or dismissal?

26. Discuss the possibilities of self-supervision by teachers.

27. In the light of an honest appraisal of your own limitations in training and experience, draw up a statement of the specific ways in which you think a competent and sympathetic supervisor could be helpful to you in your effort to become a better teacher of mathematics.

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PART III

THE TEACHING OF THE SPECIAL SUBJECT MATTER OF SECONDARY MATHEMATICS

THE JOB OF THE MATHEMATICS TEACHER

The teacher of mathematics, like all other teachers in the secondary school, is a person of whom many things are expected. His obligations are not confined to the classroom but extend along many avenues to the promotion of the effective functioning of the school and the maintenance of harmonious relations and constructive understanding between the school and the community. The sponsoring of extra-curricular activities, cooperation in maintaining a smoothly operating physical organization, participation in counseling and guidance, keeping careful records, making necessary reports promptly, and participation in worthy community interests are but illustrative of the range of demands upon the teacher. It must not be forgotten, however, that his first and foremost obligation is to teach effectively.

Teaching mathematics in the secondary schools is a task which, if seriously undertaken, will challenge the best efforts of the best teachers. It requires more than a thorough knowledge of the subject matter to be taught, though that, of course, is a *sine qua non*. It requires more, even, than a broad perspective of the field of mathematics itself and an understanding of the place and importance of mathematics in any valid scheme of general education. It demands skill in the techniques of teaching each particular topic or aspect of the subject, in developing generalized concepts, in coordinating generalizations with applications, in discriminating between essential and unimportant matters within the subject, in knowing where to place emphasis and where to anticipate difficulties, in detecting difficulties when they do occur, in sensing their precise nature, and in knowing how to help the students avoid or overcome them.

The first task of the teacher in connection with the teaching of any division, unit, topic, or aspect of mathematical subject matter is to decide just what the immediate and definite objectives are, *viz.*, which concepts or items of information the students are to gain from their study of that topic, which skills are to be mastered, which techniques and materials will be most effective in producing the desired results. Economy and clarity of learning will come only insofar as the multitude of details related to the unit are subordinated to the main issues and are organized and integrated around the few really major concepts

and skills so that these will stand out in bold relief against the background of contributory detail. Having decided what important things are to be emphasized in a particular unit, the teacher is then in a position to view the whole problem in its proper perspective and to organize and present the material of the unit in a more effective manner.

The teacher of mathematics must remain alert at all times to the six major objectives of instruction (see page 16). If these are to be attained in an effective and economical manner, the teacher must plan his work with at least five things in mind. (1) He must decide what exercises and activities will contribute most effectively to produce the desired understandings and skills. This teaching material should be selected with great care. (2) He must analyze these teaching materials carefully to anticipate the specific difficulties which the pupils are likely to encounter in attaining the objectives of the unit. (3) In order to help the pupils avoid or overcome these difficulties, the teacher must become expert in sensing the procedures and devices that promise to be specifically helpful and must learn to be adept in adjusting his procedure to the requirements of each immediate situation. Explanations and developmental discussion should be pointedly and skillfully organized. Devices and illustrations should be selected with great care. (4) A careful selection and arrangement of motivating materials must be made. The teacher must keep in mind that it is his responsibility to create, stimulate, and maintain interest in mathematics as well as to strive for proficiency in skills and the amassing of information. (5) He must give careful thought to evaluation techniques and remedial procedures.

No teacher can do a thoroughly good job of teaching mathematics unless he is willing to make a careful analysis of his job and to be guided by that analysis in making his preparations and in conducting the work of the class. The analysis of the instructional problems involved in the teaching of any topic in secondary mathematics seems to divide itself rather logically into six considerations as follows:

1. What background of experience and understanding may the student be expected to have when he begins the study of the topic?
2. What are the particular understandings or abilities which the student should acquire or strengthen through the study of the topic?
3. What activities or procedures on the part of the teacher and student will enable the student most effectively to gain these desired understandings and abilities?

4. What specific difficulties may the student be expected to encounter in his effort to acquire these understandings and abilities?
5. What specific suggestions, devices, and procedures will help the student most effectively to avoid or overcome these specific difficulties?
6. What materials and procedures related to the particular topic will best stimulate and maintain the student's interest?

It is the purpose of Part III to consider some of the more important instructional units of secondary mathematics in the perspective of the above questions.

CHAPTER XI

THE TEACHING OF ARITHMETIC

Our word "arithmetic" is derived from the word "αριθμητική," which the Greeks used to contrast the *science of number* with the *art of computing*, or "λογιστική." This distinction in terminology prevailed until the sixteenth century when "arithmetic" came to mean both the science of number and the art of calculation. In modern American parlance *numerology* is used to refer to the study of the mystic aspects of number, and *theory of numbers* refers to the scientific study of number relations, while *arithmetic* is used with the dual connotation of the study of elementary number relations and the art of calculation. Teachers of arithmetic should keep themselves constantly alert to this duality that they may avoid the tendency to overemphasize the art of calculation at the expense of the significance of number relations. Their instructional programs must place as much emphasis upon understanding of meanings, appreciation of relationships, possibility of applications, and opportunity for integration as upon proficiency in mechanical operation.

The Arithmetical Responsibility of the Secondary School. It is the responsibility of the elementary school to build up a vocabulary in arithmetic based upon familiarity with basic concepts and fundamental meanings; to strive for skill in computing with integers, common fractions, and decimal fractions; and to provide such appropriate instructional material as will enable each pupil, through vicarious experience, to understand better the role which arithmetic will play in his everyday life. In constructing its curriculum and formulating its instructional program, the secondary school is justified in assuming that the elementary school has met this educational obligation.[1] Experience and experiment, however, have shown that the secondary school must provide opportunity for the maintenance and further development of

[1] Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, *Fifteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1940), pp. 53–54.

arithmetical ability. Schorling gave a test consisting of 100 simple arithmetic tasks to 3,545 children in grades 5 through 12 in order to try to determine how well children in these grades learn some of the things we try to teach them about arithmetic.[1]

Table 4 presents a summary of his findings for grades 7 through 12. It is to be noted that one-half of the seventh-grade children made scores lower than 40, while in the twelfth grade one-half of the group

TABLE 4. NORMS FOR GRADES 7 TO 12, INCLUSIVE, ON 100 TASKS IN ARITHMETIC*

Grade	7	8	9	10	11	12
Number of pupils	638	633	335	236	230	215
75 percentile	50.6	54.2	59.8	62.3	72.2	81.9
Median score	39.8	43.8	48.6	51.7	59.0	67.0
25 percentile	30.4	35.0	39.9	42.6	46.0	54.3
Lowest score	6.0	9.0	5.0	12.0	16.0	22.0
Highest score	86.0	91.0	94.0	92.0	96.0	94.0

* Raleigh Schorling, The Need for Being Definite with Respect to Achievement Standards, *The Mathematics Teacher*, **24** (1931), p. 316.

were not able to perform satisfactorily as many as two-thirds of the tasks. The table also indicates that there was possibly a great deal of overlapping of scores, for the lowest score in each grade is well within the lowest fourth of the scores of the seventh grade and the highest score in each grade is within the highest fourth of the twelfth grade. In summarizing the findings of this study Schorling states that

1. Although all these processes have been taught in most schools, the degree of mastery at the beginning of the eighth grade is very low and there is little subsequent increase in ability in computation.

2. In spots where increase does appear it may be due to greater maturity or a higher selection of students.

3. Even in the senior high school, mastery is very low. In the tenth grade there are only 51 things out of 100 to which half of the children responded correctly; half of the children know half of the tasks. In the eleventh grade we get 60 tests at the 50% level and in the twelfth grade we get 72 things. That is, at the end of the senior high school half of these children know three-fourths of the material.

4. Further evidence of low mastery is reflected in the median and quartile scores. To be sure, there is a steady march upward in the medians; but it is

[1] Raleigh Schorling, The Need for Being Definite with Respect to Achievement Standards, *The Mathematics Teacher*, **24** (1931), 311-329.

not so rapid that it may not be due entirely to either maturity, selection, or a combination of these two factors. Note that the median score of the twelfth-grade group is only 67; that is to say, one-half of the twelfth-grade pupils individually did less than about half of the test.

5. The highest score agrees with much available evidence of enormous overlapping. For example, while the median score in the twelfth grade is only 67, there is a score as high as 86 in the seventh grade. Recently we found a country boy who did 93 of these tasks.

6. One of the crucial problems of the senior high school, though little is said about it, appears to be that of computation.[1]

The results of a contemporaneous but independent study reported by Bridges[2] strongly support the findings given by Schorling. In a study of the Mastery of Certain Mathematical Concepts by Pupils at the Junior High School Level, Butler[3] found that the average pupil probably enters the junior high school with a mastery of not more than roughly one-third of those mathematical concepts determined as basic by Schorling.[4]

The impact of World War II served to give strong emphasis to the facts presented in the above studies. The educational literature of the war and the immediate postwar periods is steeped in individual comment and committee reports presenting the need for a more adequate program in arithmetic. While much of this writing is characterized by an intoxicated interest in the emergency needs of the military inductee, yet there is present a constant undercurrent of emphasis on the sobering parallel between such needs and those of normal civilian life.

Facts such as those just cited emphasize the responsibility of the secondary school to provide opportunity for continued effort in arithmetic. It does not necessarily follow, however, that the arithmetic of the secondary school should be taught as a separate topic. The seventh grade does have the specific responsibility of the extension of decimal fractions to percentage and the development of those concepts and skills essential to the understanding and intelligent use of per-

[1] *Ibid.*, pp. 316 317.

[2] W. A. Bridges, "Mathematical Ability of Pupils Entering the Junior High School," unpublished M.A. thesis (Nashville, Tenn.: George Peabody College for Teachers, 1931).

[3] Charles H. Butler, Mastery of Certain Mathematical Concepts by Pupils at the Junior High School Level, *The Mathematics Teacher*, **25** (1932), 165.

[4] Raleigh Schorling, "A Tentative List of Objectives in the Teaching of Junior High School Mathematics" (Ann Arbor, Michigan: George Wahr, 1925), pp. 101–102.

centage as a fundamental arithmetical tool. The major mathematical obligation of the seventh and eighth grades, however, is to strengthen and increase the working vocabulary of arithmetical terms, to effect a clearer understanding of basic principles, to develop further facility in fundamental skills, and to emphasize the abstraction of arithmetical processes to life situations.

The ninth grade should continue the emphasis on accurate arithmetical work. In addition to the pure computation involved in the evaluation of formulas and in the solving and checking of algebraic equations, the algebra of the ninth grade provides for the extension of the number concepts to include literal numbers and directed numbers. Once these concepts have been developed there is the need for the generalization of the fundamental operations of arithmetic to these new numbers and to the algebraic expressions they introduce. A further extension of the number concept to include irrational numbers and imaginary numbers is accomplished in the later work of the senior high school and junior college.

The measurement problems of geometry and trigonometry offer still further opportunities for emphasis on the fundamental skills of arithmetic. Although the instructional material of the junior high school offers a greater abundance of situations which are more nearly numerical in nature, the responsibility for increasing accuracy and facility in numerical computation does not end here as the pupil passes on to higher levels of instruction. This responsibility, as that of building up a progressive increase in the pupils' understanding of basic concepts and appreciation of arithmetical applications, extends even into the junior college.[1] At these higher levels of instruction, however, more careful and detailed attention should be given to the extension of the number system and the development of the fundamental principles and processes of approximate computation.

The Nature of Approximate Numbers.[2] All *exact* numbers result from counting and from applying the fundamental processes to counted

[1] H. Glenn Ayre, An Analysis of the Performance of College Freshmen on Arithmetic, *The Western Illinois State Teachers College Quarterly*, **19**, No 2 (1939).

C. C. Richtmeyer, Functional Mathematical Needs of Teachers, *Journal of Experimental Education*, **6** (1938), [illegible] 398.

Raleigh Schorling, The Need for Being Definite with Respect to Achievement Standards, *The Mathematics Teacher*, **24** (1931), pp. 317–320.

[2] The best single reference on approximate computation is probably Aaron Bakst, Approximate Computation, *Twelfth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publication , Teachers College, Columbia University, 1937).

quantities. When we say that there are 35 children in the sixth grade, that Jane receives $2 each week as her allowance, or that six eggs make $\frac{1}{2}$ dozen eggs, we mean *exactly* what we say. The counting process sets up a one-to-one correspondence that establishes this exactness. On the other hand, numbers which are estimated results, even though based on counting, as in the case of the census, or which record the data resulting from measurement are *approximate* numbers. This is true whether we are measuring distance, direction, temperature, or what not. If we say that Jack is 5 feet 6 inches tall, we can only mean that, according to our measuring technique, his height is nearer to this measurement than it is to any other. No measurement can be more precise than the precision of the measuring instrument or more accurate than the relative accuracy of the observation made.

Approximate numbers may arise also from certain mathematical processes which require an infinite number of steps of which only a finite number can be performed. The extraction of certain roots, such as $\sqrt{2}$ and $\sqrt[3]{6}$, the expansion in decimal form of certain nonterminating fractions such as $\frac{2}{3} = 0.666\dot{6}$ and $\frac{1}{7} = 0.\dot{1}4285\dot{7}$, and the evaluation of transcendental numbers such as $\pi = 3.14159 \ldots$ and $e = 2.71828 \ldots$ are examples of such processes. It should not be inferred from the above remarks that irrational numbers or fractions are necessarily approximate. Whether rational numbers or irrational numbers are exact or approximate depends upon the interpretation of the data involved. If we have a square whose side is exactly 1 unit in length, then $\sqrt{2}$ is the exact length of its diagonal. The actual measurement of this length would, of course, be only approximate.

It should also be stated that exact numbers may be fractional while the approximation is integral; for example, if lemons are selling at 45 cents a dozen, the 23 cents one pays for $\frac{1}{2}$ dozen is an approximation to the $22\frac{1}{2}$ cents which is the exact price. The context in which any given number is used will frequently play an important part in determining whether it is to be regarded as an exact or an approximate number. An individual who states that he purchased 2 pounds of butter is using the number "2" in an exact sense if he means two 1-pound cartons, whereas the actual weight of the butter only approximates 2 pounds. Furthermore, the use of formulas in practical computations frequently gives rise to approximate numbers either because of the approximate nature of the formula itself (as in the case of the formula $A = 3.14r^2$, or $A = \frac{22}{7}r^2$, for the area of a circle and formulas resulting from scientific experimentation) or the use of approximate

data in an exact formula (as would result in substituting measurements in the formula $A = lw$ for the area of a rectangle).

Although facility and accuracy in computation with exact numbers is both a desirable and a necessary goal of arithmetical instruction in the elementary school, the secondary school should stress the "exercise of common sense and judgment in computing from approximate data, familiarity with the effect of small errors in measurements, the determination of the number of figures to be used in computing and to be retained in the result, and the like."[1] There is no justification whatsoever, for example, in stating that the circumference of a circle whose radius is given as 3 inches is 2(3.1416)(3) = 18.8496 inches. The measurement of the circumference can be no more precise or accurate than the measurement of the radius used in finding the circumference. It is then very important that the teacher and pupil understand certain fundamental criteria for judging approximativeness[2] and rules for computation with approximate data.

Criteria for Judging Approximativeness. The three principal criteria for judging approximativeness are:

1. *The Position of the Decimal Point.* A number may be said to be correct to within a certain unit (*e. g.* to units, tenths, hundredths, etc.). The distance from the earth to the sun is usually given as 93,000,000 miles. Here the unit of measurement is 1,000,000 miles, and the measurement is considered correct to the nearest unit. The world record for the 100-yard dash is 9.3 seconds. Here either a second or one-tenth second may be taken as the unit of measurement. The observation is said to be correct to tenths of a second. In such cases as these the number of decimal places in the observation proves to be a criterion for judging approximativeness.

2. *The Number of Significant Digits.* Our decimal system of numeration is definitely characterized by the fact that the significance of any particular digit in a number is determined by the position it occupies. In the number 333.3 each three denotes a value one-tenth as large as the one on its left and ten times as large as the one on its right. Thus the number 333.3 is given to four significant digits since each three has a specific relative significance in the make-up of the number. If we consider the numbers 303, 3.03, and 33, it is evident that the pres-

[1] National Committee on Mathematical Requirements, "Reorganization of Mathematics in Secondary Education" (Boston: Houghton Mifflin Company, 1923), p. 7.

[2] By "approximativeness" is meant the closeness with which the approximate number approaches the exact number.

ence or absence of the zero affects the relative magnitudes of the threes. In the first two numbers the three on the extreme left is of a magnitude 100 times as great as that of the three on the extreme right, while in the third number the three on the left is only 10 times as great in magnitude as the three on the right. Now consider the numbers, 33, 3.3, 0.033, and 330. It is just as evident that the presence or absence of the zero does not affect the relative magnitude of the threes. In each case the magnitude of the three on the left is 10 times that of the three on the right. In the number 330, as in 0.033, the zero serves merely as an aid in placing the decimal point to distinguish 330 from 33, 3.3, 0.033, 33,000, etc., in which case it is not considered as a significant digit. Each of the numbers 303 and 3.03 is given to three significant digits, while 33, 3.3, 0.033, and 330, where zero merely helps to place the decimal point, are each to two significant digits. Similarly the 93,000,000 and 9.3 given above are each to two significant digits.

The value of π to seven significant digits is

(1) $$\pi = 3.141593$$

This value is very frequently stated as

(2) $$\pi = 3.1416$$

In (2) the value of π given in (1) has been *rounded off*. The rules usually given for rounding off numbers may be stated as follows:

1. If a whole number, given to a certain number of significant digits, is to be rounded off to a stated number of significant digits, the digits that are to be dropped should be replaced by zeros. In the case when the digits that are to be dropped are located to the right of the decimal point, the use of the zeros is not correct.

2. If the first digit on the left of those that are to be dropped is 5, 6, 7, 8, or 9, then the first digit on the extreme right of the number, which is to be retained, should be increased by unity. This process is known as "forcing the digit." If the first digit on the left of those that are to be dropped is 0, 1, 2, 3, or 4, then no change is made in the digits retained.

3. If after the forcing, the significant digit on the extreme right is 5, a bar should be placed over it in order to indicate that if there should be a necessity to drop this 5, the next digit on its left should remain unchanged. For example: 3,464,832 is rounded off to $3,46\bar{5},000$ and this is rounded off to 3,460,000.

If we apply the above rules to round off each of the numbers 296 and 303 to two significant digits, we obtain in each case 300. The zero

on the left thus becomes a significant digit and is underscored to indicate this fact. Whether the zeros to the right of all nonzero digits of an approximate number are significant or not must be determined from an analysis of the situation which produced the number. In the case of a measurement the significance of such zeros can be determined if the precision of the measurement or the unit of measurement is known. In the case of rounded numbers it can be determined only by reference to the numbers from which the rounded numbers were obtained. If only a number, such as 39,000, is given and nothing is known about what it represents or how it was obtained, then there is no way of deciding whether any of the zeros are significant or not. However, all zeros would be significant in 390.00 meters since the two zeros to the right of the decimal point would not be used as an aid in placing the decimal point but to signify that in the application of the specified unit of measurement *no* quantities were found to occupy the two places to the right of the decimal point. Similarly, the zeros in $390 and $390.50 would be significant, except in the case where these amounts were given as mere estimates. Zero is significant whenever it is used other than as a mere placeholder to assist in the proper placement of the decimal point.

A very effective method of indicating significant digits, particularly in the case of computation with large numbers, is the scientific form of notation. *A number is said to be written in the scientific form of notation when it is written as the product of a number between* 1 *and* 10 *and an integral power of* 10. Any significant zeros would be excluded from the power of 10. For example, the volume of the sun is 1,300,000 times the volume of the earth, while Mars is only 0.150 times as large as the earth. If we use S, M, and E to represent the volumes of the sun, Mars, and the earth, respectively, the above statement may be written in scientific notation as follows:

$$S = 1.3 \times 10^6E, \qquad M = 1.50 \times 10^{-1}E$$

We may now summarize the basic considerations with reference to significant digits as follows:

1. Any nonzero digit is significant unless it indicates unwarranted precision in a measurement or unwarranted approximativeness resulting from computation with approximate numbers. In the latter case the significance is to be determined by the rules of the specific type of computation. Generally speaking, the result of any computation with approximate data can contain no more significant digits than the smallest number of significant digits contained in any number used in the computation.

2. Zeros occurring between significant digits are significant.

3. Zeros to the right of a nonzero digit are significant only when underscored or when they are to the right of the decimal point.

4. Zeros used merely for placing the decimal point are not significant.

5. The significant digits of an approximate number include the first nonzero digit on the left of the number, the digit which shows the precision of the number, and all digits in between these two digits.

Illustration. The following numbers are all correct to five significant digits: 3 2674, 30207, $3126\underline{0}$, 312.67, 3126.0, 0.031267, 0.000031267, 0.00030067, 2400.0, $3000\underline{0}$.

3. *Precision and Accuracy.* Although precision and accuracy are distinctly different as criteria for the measure of approximativeness, they can be most effectively discussed when contrasted with each other. Measures may be precise to within certain specified units as 1,000,000 miles, 1 mile, 1 second, $\frac{1}{10}$ of 1 second, etc. Similarly, numbers may be precise to units, tenths, hundredths, etc. On the other hand, a measure or a number may be accurate within a certain per cent of error or a certain number of significant digits.

The most effective measures of both precision and accuracy are in terms of the errors involved. The maximum error (positive or negative) made in any approximation is defined as follows: "If an approximate number is given as correct to k significant digits, then its error is at the most equal to ± 0.5 of a unit in the kth place, counting from the left to the right."[1]

The *upper* and *lower limits* of the true value of any approximation are obtained by adding this maximum error to, and subtracting it from, the approximate number. The maximum *apparent error* involved in any approximation made to k significant digits is thus seen to be 0.5 of a unit whose magnitude is determined by the kth place of the approximation. For example, the upper and lower limits of the true value of the distance from the earth to the sun are 93,500,000 and 92,500,000 miles, respectively, so that the limit of the apparent error is 500,000 miles.

In the measure of time for the foot race the upper and lower limits are, respectively, 9.35 and 9 25 seconds, and the limit of the apparent error is seen to be 0.05 second. *The precision of a measure or a computation is evaluated in terms of the apparent error.*

The two measures, 93,000,000 miles and 9.3 seconds, are both correct to two significant digits. The per cent of accuracy of a measure

[1] Bakst, *op. cit.* p. 124.

or a computation is determined by the *relative error* involved, *viz.*, the ratio of the apparent error to the approximate number. The relative errors, to two significant digits, in each of the above cases are, respectively,

$$\frac{500{,}000 \text{ miles}}{93{,}000{,}000 \text{ miles}} = \frac{0.5}{93} = 0.0054$$

and

$$\frac{0.05 \text{ second}}{9.3 \text{ seconds}} = \frac{0.5}{93} = 0.0054^{*}$$

The per cent *of error* in each case is thus seen to be the same, $\frac{1}{2}$ of 1 per cent. An approximation might be far less precise than another and yet be much more accurate. For example, suppose we have the two measurements 0.000341 inch and 1,256 feet. The first measure is much more precise since its maximum apparent error is 0.5 of a millionth of an inch, while that of the second measure is 0.5 of a foot. The first measure is correct to three significant digits and the second to four; the relative error is

$$\frac{0.0000005}{0.000341} = \frac{0.5}{341} = 0.0015$$

in the first case and $0.5/1{,}256 = 0.0004$ in the second. Thus the per cent of error is 0.15 per cent in the first measurement and 0.04 per cent in the second; in other words, the second measure, although far less precise than the first, is about four times as accurate. *The accuracy of a measure or a computation is evaluated in terms of the relative error or per cent of error made.*

When common fractions are used in giving approximate data, the denominator of the fraction states the unit of precision used in making the measurement, while the numerator indicates the number of significant digits to which it is read. The unit of precision in each of the following measurements is one-fourth inch: $\frac{3}{4}$ inch, $6\frac{1}{4}$ inches, and $34\frac{3}{4}$ inches. The number of significant digits in each is $\frac{3}{4}$ inch, one; $6\frac{1}{4}$ inches $= \frac{25}{4}$ inches, two; and $34\frac{3}{4}$ inches $= \frac{139}{4}$ inches, three. While $6\frac{1}{4}$ inches has the same numerical value as $6\frac{4}{16}$ inches, there is a great deal of difference in the precision of the two measures. Similarly there is definite significance to be attached to a measure of $5\frac{5}{8}$ inches as contrasted to one of 5 inches. The maximum apparent error in $5\frac{5}{8}$ inches is $\frac{1}{16}$ inch, while in 5 inches it is $\frac{1}{2}$ inch.

Computation with Approximate Data. In any computation involving approximate data, the result can never be any more precise or accurate than any of the data used. While the rules for such computation may be stated in several different forms, probably the two most satisfactory rules are:

1. *In the addition or subtraction of approximate numbers of the same degree of* precision, *perform the operation and retain the result to the same degree of precision. If the numbers are not of the same degree of precision, first round all numbers to the same unit of precision and then proceed with the computation.*

2. *In the multiplication or division of approximate numbers of the same number of* significant digits, *perform the operation and then round the result to the same number of significant digits. If one approximate number has more significant digits than the other, first round the more accurate number so that it has only one more significant digit than the less accurate number. Perform the operation and then round the result so that it contains the same number of significant digits as the less accurate number.*

With some sacrifice in economy of effort but no essential difference in significance of results some writers prefer, for very elementary work with approximate numbers, to simplify these rules to read:

In any computation with approximate numbers first perform the required operation with the given numbers just as if they were exact numbers, then round the results: (1) *in addition or subtraction, to the same unit of precision as the least precise number used;* (2) *in multiplication or division, to the smallest number of significant digits that occur in any number used.*

The rule for multiplication may, of course, be extended to control the results obtained in raising an approximate number to any given power, and the rule for division to the extraction of indicated roots.

The intelligent use of these rules combined with care in the statement of original data will produce results that can be justified as the best possible results to be obtained from the given data.

Mensuration and Denominate Numbers. It has been said "that the need for rules in using numbers first arose in applying them to measurements which can never be exact."[1] This statement depicts the very close relationship that exists between mensuration and arithmetic. The facts of measurement are portrayed through the language

[1] Lancelot Hogben, "Mathematics for the Million" (New York: W. W. Norton & Company, 1937), p. 66.

of number. It is at times hard to tell where geometry ends and arithmetic begins in the study of mensuration problems, a fact which is the basic reason for placing a great deal of responsibility on the secondary school for the arithmetic of measurement.

> No arithmetic course is complete without opportunities to learn how mankind has gradually evolved a system of measures and a technique for measuring, to acquire information about the measures in common use, to develop some skill in the use of measures and in estimating quantities and magnitudes without measuring, to learn something of the degree of accuracy normally present in measurements, and to gain an appreciation of the role of measurement in modern life.[1]

Every pupil should have the opportunity to learn the distinguishing characteristics and distinctive techniques of both direct and indirect measurement. The major emphasis in the junior high school should be on direct measurement. This emphasis should shift to indirect measurement with the development of trigonometry, starting in an elementary way with the numerical trigonometry of the junior high school and receiving its major treatment in the senior high school and junior college. The pupil should become thoroughly familiar with the use of various measuring instruments, and the teacher should not be too ready to assume such familiarity even with the simplest instruments of linear measure. Not only should opportunity be provided for making many measurements with scales based on different units, including those of the metric system, but also attention should be given to the practice of estimating such measurements, especially lengths, heights, areas, and volumes. Careful checks should be made of such estimates, not only for the purpose of increasing the accuracy of estimate but also to emphasize the approximate nature of measurement. Careful checking, one against another, of measurements of the same object made with scales based on different units will also aid in developing a consciousness of the approximate nature of measurement and a feeling of the relationship among the units in the different systems.

The pupil should have his attention called to the three distinct types of errors involved in making any measurement. *Constant errors* are those that are due to the measuring instrument, to constant atmospheric conditions, or to any other constant element that might affect

[1] R. L. Morton, "Teaching Arithmetic in the Elementary School," Vol. III, "Upper Grades" (New York: Silver Burdett Company, 1939), p. 282.

the determined result. A yardstick might be too long, a steel tape might be contracted or expanded because of changes of temperature, the individual making the measurement might constantly err in the application of the instrument, etc. It is the presence of such errors as these that makes it necessary to calibrate the measurements taken in record trials, such as balloon ascensions, automobile races against time, etc. *Mistakes* are errors which are due to the carelessness or inexperience of the person making the measurement. An individual might make a protractor reading for the size of an angle as 38 degrees instead of 42 degrees; he might read a length as $6\frac{1}{8}$ inches when it was $6\frac{3}{16}$ inches; etc. Such errors as these are of the type that practice can help to correct. *Accidental errors* are those that are due to unknown or uncontrollable conditions, a sudden gust of wind, position of observer making a reading, sudden atmospheric changes, etc. Such errors as these are handled through the application of *statistical measures.*

The secondary-school pupil should thus have the opportunity of learning how to make a frequency distribution of scores and how to calculate the *arithmetical average* and *median of the distribution.* He should understand that the average is the best score to use to represent all scores unless there are a few which are far removed from the rest of the scores, in which case the median gives a more typical representation. He should also know how to use the bar graph, circle graph, and broken-line graph to represent such distributions. To make these graphs satisfactorily one must know the fundamentals of drawing to scale, which implies the selection of appropriate units of measurement and adjusting the given scores to the chosen unit. This necessitates a careful program of intelligent instruction that the pupils may know just what is involved in the choice of the unit and how to proceed in making a scale drawing which analyzes the given data in terms of the unit.

In the emphasis upon the approximate nature of measurement one should not forget that such approximation can often be made with high degrees of accuracy and precision. The pupils' attention should be called to the dependence of modern industry upon precise and accurate measurement. Illustrations should be given of such cases as those in which measurements can be made accurate to one-millionth of an inch or of experiments where it has been possible to detect variations as minute as 1 part in 100,000,000.[1]

[1] William Betz, The Teaching of Direct Measurement in the Junior High School, *Third Yearbook of the National Council of Teachers of Mathematics* (New York:

Because of its importance in scientific laboratories, the metric system should be introduced and developed to the extent that every secondary-school pupil would be familiar with its more fundamental aspects. No great emphasis should be placed on equivalence between units in the metric and English systems. Some would insist that the only emphasis that should be given is to the approximate value of the basic unit of the metric system (1 meter = 39.37 in.).

Common sense should predominate in the treatment of denominate numbers. There are only a few scales of measurement which are national in extent, such measures as those of height, weight, distance, area, and volume are examples. Every school situation should emphasize the importance of these units of measure and the interrelations of such units. Everyone should know that there are 12 inches in 1 foot, 3 feet in 1 yard, and 5,280 feet in 1 mile. It is doubtful that everyone should know that $5\frac{1}{2}$ yards = $16\frac{1}{2}$ feet = 1 rod. Everyone should know what 1 square foot means and why 1 square foot = 144 square inches. The teacher should become familiar with all such basic measurements and point his instruction to intelligent mastery on the part of the pupils.

There are certain aspects of measures and denominate numbers which become somewhat regional in nature. For example, the rural child is much more likely to have need for the relation between rods, yards, and feet than is the urban child; the Southern child for measures of cotton than the Northern child; the Californian for measures of fruit than the New Englander; the Middle Westerner for measures of wheat and corn than the Easterner, etc. Such facts as these place a definite responsibility upon the classroom teacher to discover such regional differences and make provision for them in his instructional program.

Percentage. Without some fundamental understanding of percentage it would be impossible to perform such simple and universally useful activities as computing interest, determining discounts, and making simple routine comparisons of quantitative data. Even the intelligent perusal of the daily paper requires at least an elementary understanding of percentage. Aside from the fundamental processes of arithmetic, percentage probably has more widespread and direct applications to the ordinary affairs of all people than any other special mathematical topic. Every reasonable effort should, therefore, be made to bring the teaching of percentage to such a degree of effectiveness that at least a

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Bureau of Publications, Teachers College, Columbia University, 1928), pp. 188–190.

majority of the students who come into the upper grades of the secondary school will have some genuine functional understanding of this important topic. Evidence that this has not been generally characteristic of instruction in percentage may be found in such studies as those presented by Brueckner,[1] Edwards,[2] and Schorling.[3]

Brueckner found the most common errors, other than mere computational errors, to be:

1. In changing decimals to per cents:
   a. Drops the decimal point and annexes the % symbol
   b. Copies number and annexes % symbol
   c. Changes decimal to equivalent fraction and annexes % symbol
   d. Omits integers in mixed decimal and changes decimal to per cent
2. In changing per cents to decimals:
   a. Merely drops % symbols in answer without changing to hundredths
   b. Moves the decimal point to the left incorrectly
   c. Inserts unnecessary zeros
   d. Drops % symbols and annexes zeros
3. Changing common fractions to per cent:
   a. Lacks knowledge of per cent equivalents
   b. Adds % symbol to fraction without changing its form
   c. Divides numerator of fraction by denominator but fails to carry work to hundredths
4. Changing fractions to hundredths and to per cents:
   a. Merely copies numerator and writes as a two-place decimal
   b. Copies entire fraction and writes as hundredths
   c. Multiplies numerator by denominator
   d. Errors in changing fraction to hundredths
5. In finding a per cent of a number:
   a. Adds % symbol to answer
   b. Divides the numbers
   c. Errors in changing per cents to decimals
6. In finding what per cent one number is of another:
   a. Divides base by percentage
   b. Fails to express quotient as per cent
   c. Multiplies base and percentage
   d. Errors due to faulty manipulation of decimals

[1] Leo J. Brueckner, "Diagnostic and Remedial Teaching in Arithmetic" (Philadelphia: John C. Winston Company, 1930), pp. 241–257.

[2] Arthur Edwards, "A Study of Errors in Percentage," *Twenty-ninth Yearbook of the National Society for the Study of Education* (Bloomington, Ill.: Public School Publishing Company, 1930), pp. 621–640.

[3] Raleigh Schorling, The Need for Being Definite with Respect to Achievement Standards, *The Mathematics Teacher*, **24** (1931), 311–320.

7. In finding a number with a per cent given:
   a. Multiplies numbers representing rate and percentage
   b. Divides rate by percentage
   c. Divides percentage by rate, disregarding decimal form
   d. Errors in manipulation of division of decimals
   e. Subtracts numbers representing rate and percentage
   f. Inability to manipulate fractions after changing rate to equivalent fraction[1]

Percentage is naturally an extension of decimal fractions. Whereas in the general treatment of decimals there were different units to be considered, in the study of percentage there is concentration upon but one, *viz.*, hundredths. The operations involved are not new operations but merely new applications of familiar practices. In working with both common and decimal fractions, the pupils have had to find a fractional part of a given number, what fractional part one number is of another (the ratio between two numbers), and to find a number when a fractional part of it is given. These are recognized as the customary three cases of percentage. Why label them with such formality in percentage when they have been treated more naturally at previous stages of arithmetical instruction? The pupils should be given the opportunity to find applications of percentage and should be shown the fundamental relationships involved. They should be brought to realize that the one fundamental principle that underlies all percentage problems is: *base times rate equals percentage*, or in formula form

$$br = p$$

This formula should be used as a basis for integration of thinking in percentage. The pupils have used the formula for area, $A = lw$, and they should see the similarity of structure in the two formulas. In fact, such formulas offer fine opportunities for the generalization of arithmetical processes to algebraic processes, just as measurement offers opportunities for integration of arithmetic and geometry.

The proper use of the basic formula for percentage will aid in closely relating the three cases; in fact, they become merely three aspects of the same problem. Such instruction will point to fundamental understandings rather than mere mechanical reactions to type rules. Care should be exercised by the teacher not to place too much emphasis, in the beginning, on the use of terms. Every effort should be exerted

[1] From "Diagnostic and Remedial Teaching in Arithmetic," by Leo J. Brueckner, John C. Winston Company, Philadelphia, 1930, pp. 255 256.

to stress basic concepts and to develop an understanding of and appreciation for important relationships.

Socialized Arithmetic. The teaching of arithmetic has two major responsibilities: (1) the building up of an appreciative understanding of our number system and an intelligent proficiency in the fundamental processes which history has provided as a means for the efficient use of this system; (2) the socialization of number experiences that they may contribute to the improvement of the common thinking practices of the race. The first responsibility is largely that of the elementary school, while the second is largely that of the secondary school.

There is no implication in this statement that the arithmetic of the elementary school should overemphasize its computational aspects to the neglect of its informational and socio-utilitarian functions. At this level of instruction the pupil should have many rich experiences in the uses and applications of the fundamental concepts and processes of arithmetic in order that he may have more intelligent insight into and appreciative understanding of their environmental significance. It should be emphasized, however, that these experiences should be provided as situations which call for the application of arithmetical concepts and processes which have been systematically studied rather than as projects which merely suggest certain concepts and processes to be promiscuously presented.

There have been a great many efforts to determine those topics of arithmetic which have definite social value. Most of these attempts have approached this problem from what has been called the "consumer's point of view" and have tried to ascertain what the adult uses of arithmetic are. The usefulness of such job-analysis technique, however, has definite limitations. A certain plausibility does attach to curricular content based solely on this type of analysis, but this very plausibility is likely to be specious and misleading since its basis is only a partial basis. Social utility is indeed a valid criterion, but it by no means follows that it should be the only criterion to be considered. A fundamental oversight in such procedure is that our educational program should be determined in the light of what the future adults of our country *ought* to be able to use rather than what the present adults *do* use. Furthermore, as a criterion for determining what arithmetic should go into the elementary curriculum,

. . . the criterion of social utility as it is ordinarily determined by a survey of the uses of arithmetic by adults is only a partial criterion. There are other ways of determining value. Some of these are more or less obvious, such as the abilities of pupils, their social status, and the likelihood of their

going on to the high school, and so on. These considerations obviously have reference to the individual.[1]

In the secondary school the situation is somewhat different. By the end of the first half of the seventh grade the pupil has had the opportunity to become acquainted with the fundamental processes of arithmetic as applied to integers, common fractions, and decimal fractions including percentage. Although there will be continued need for practice in these processes, the principal arithmetical responsibility of the secondary school is to provide opportunities for the quantitative interpretation of social environment. There is no longer an instructional hierarchy of arithmetical skills that must be observed. The material can be organized in large units of instruction which emphasize for the pupil the importance of arithmetic in the better understanding of many of his normal daily experiences.

Arithmetic should make a vital contribution to the intelligent consideration of various aspects of business, consumption, production, government, and social relationships which lend themselves to quantitative study and analysis. The real meaning and significance of profit and loss, the responsibilities and difficulties of the retailer and the wholesaler, and the relationships of the home and business establishments are part of the field of arithmetic . . . In no other subject is there the opportunity to teach the pupils the considerations that should underlie the investing and saving of money, the methods of investing money, the ways of sending money in safety from one place to another, convenient ways of carrying money, and many other problems dealing with the quantitative aspect of social relationships.[2]

Not only will the alert teacher of secondary mathematics accept the responsibility of unveiling to the pupil something of the social value of arithmetic, but he will also recognize and make use of the fact that, in many instances, the full significance of social progress is impressively portrayed through the medium of number language and computational techniques. A suggestive outline for the construction of such a quantitative unit on social progress is the following:

[1] B. R. Buckingham, The Social Value of Arithmetic, *Twenty-ninth Yearbook of the National Society for the Study of Education* (Bloomington, Ill.: Public School Publishing Company, 1930) Part I, Chap. 2, p. 38. Quoted by permission of the Society.

[2] Leo J. Brueckner, A Critique of the Yearbook, *Twenty-ninth Yearbook of the National Society for the Study of Education* (Bloomington, Ill.: Public School Publishing Company, 1930), Part II, Appendix, p. 690. Quoted by permission of the Society.

The Story of Transportation

1. The Pony Express.

How long has it been since the days of the Pony Express? What were the costs and profits associated with its operation? How do average times for making certain distances compare with present-day averages? How do fares compare? How do state populations of that period compare with the most recent census? Compare postage rates.

2. The Stage Coach.

Continue comparisons indicated above. Compare pony express and stage coach transportation.

3. Railways.

Continue above comparisons. Study distribution and source of railroad income. Costs of operation. Costs of construction. Indirect measurement in surveying. Comparative study of causes of accidents with emphasis given to "Stop! Look! Listen!" Freight and passenger service.

4. Automobiles and Busses.

Same general outline as for railways. Upkeep of automobile. Automobile and travel insurance. Highway construction. Planning trips. Compare with railways.

5. Water Transportation.

Freight and passenger service. Oceans, rivers, and canals. Compare with railways and busses.

6. Air Travel.

Rapidity of development. Mail and transport. Compare with other forms of travel.

Business Arithmetic. A somewhat extreme form of socialized arithmetic that is being tried out in some schools is a type of consumer's arithmetic. This is primarily a terminal course designed for the twelfth grade of the senior high school and in some cases for the junior college. It has been called "business arithmetic" (or "mathematics"), "consumer arithmetic" (or "mathematics"), and "social-economic mathematics." Usually it is for those students who do not plan any further work in mathematics but who need additional business training. Although the primary emphasis is usually arithmetical in nature, it seems to be customary to assume that the prospective pupil has completed ninth-grade mathematics, *viz.*, the equivalent of one year of algebra. Experiment with this type of course is fairly recent, and only tentative suggestions can be made as to the contents of such a course. A suggested outline is as follows:

1. Measurement and computations; degree of accuracy; approximate numbers; significant figures; short ways of multiplying and dividing; logarithms; computing machines.

2. Such review of the fundamental operations with fractions, decimals, and percentages as may be necessary; practice in organizing and presenting problems that may involve only arithmetic, but that are more difficult than those of the ninth grade.

3. The simpler ideas of statistical methods, considered early in the course so that the subsequent topics may be subjected to mathematical analysis as far as possible.

   a. Construction of various types of graphs, including those with logarithmic (or ratio) scales on one axis.

   b. Frequency tables; various types of averages and means, scatter diagrams.

   c. Measures of central tendency and dispersion; correlation.

4. Index numbers. Their construction and use in connection with commodity prices, real wages, cost of living, business cycles, etc.

5. Household budgets; the percentages spent on food, clothing, and shelter at various levels of income; cooperative enterprises.

6. Installment buying; reasons for apparent high rates of interest; influence on business cycles: advantages and disadvantages.

7. Investments: stocks, bonds, mortgages, investment trusts; banking procedures; periodic accumulations and payments; cost of home owning; annuities.

8. Insurance: home, fire, theft, property, accident, etc.

9. Taxation: property, sales, income, direct and indirect; the cost of government.

10. Topics involving national policies, such as crop control, price fixing, social security, tariffs, foreign exchange, distribution of national income, etc.[1]

Verbal Problems. Any treatment of arithmetic that is truly social in nature necessarily places an emphasis on the solving of verbal problems. An analysis of the literature on problem solving in arithmetic exhibits the following as the principal sources of student difficulty:

1. Computation
2. Lack of reasoning ability
3. Poor procedure or complete absence of systematic attack
4. Difficulty in selecting the process to be used
5. Failure to comprehend the meaning of the problem
6. Inefficient reading habits
7. Vocabulary difficulties
8. Short attention span or mental laziness
9. Inability to select essential data
10. Carelessness in transcribing
11. Poor eyesight and other physical defects

[1] Joint Commission, *op. cit.*, p. 117.

There have been a good many experiments made in the effort to determine a best method for teaching problem solving. Several methods have been tried, *viz.*,

1. *The analysis method,* in which effort is made to have the pupil systematically analyze the problem by requiring him to go through the sequence of steps: (*a*) What is given? (*b*) What is required? (*c*) What operations are to be used? (*d*) Estimate the answer. (*e*) Solve the problem. (*f*) Check your answer.

2. *The method of analogies,* in which the pupil is given a simple oral problem similar to the difficult written problem. The assumption is, of course, that the oral problem can be solved and that the pupil can be brought to see the analogy between the two.

3. *The method of dependencies,* in which the pupil is taught to recognize fundamental dependencies that exist within a given problem. This procedure can become a very important step in the analysis method.

4. *The graphic method* in which some diagrammatic scheme is used to aid the pupil in determining the fundamental relationships existing between the known and the unknown.

No single method stands out as the one best method; each has been productive of fairly good results. It is evident, however, that systematic attention to problem solving is worth while and that the results produced justify the time and effort expended. The implication would then seem to be that it is the responsibility of the teacher to make the attempt to analyze the difficulties present in his own teaching situation and shape the program of teaching verbal problems to fit these needs. He should become familiar with all the methods suggested above and should make use of those which seem to fit best the requirements of each immediate situation. There are a few fundamental assumptions that might be postulated as a point of departure:

1. Since the problems are written problems, the pupils must be able to read.

2. The pupils must be able to use the fundamental processes of arithmetic.

3. The pupils must be able to distinguish between essential and unessential data; hence some definite instructional program may be necessary to aid the pupils in so doing.

4. The pupils must be able to distinguish the known from the unknown. This may also be a place for needed instruction.

5. The pupils must be able to sense such relationships as may exist among given data, and between given data and the required information.

6. The pupils must be able to translate verbal expressions into mathematical symbols. This is one of the most important of all techniques neces-

sary for an intelligent approach to the solving of verbal problems. The teacher should place a great deal of emphasis upon the translation of the English statement of problem situations into their symbolic statement. Attention should be called to the simplification of operation which results from the use of significant symbolism.

7. Independence and confidence are desirable attributes of character and distinct aids in problem solving; hence pupils should know how to estimate and check answers.

Exercises

1. What are the attainments which the Joint Commission assumes as those which "may be regarded as the normal mathematical equipment of the American pupil who has satisfactorily completed the work of the sixth grade"?

2. Contrast this assumption with that made by the National Committee on Mathematical Requirements in its 1923 report.

3. Do you consider the assumption of the Joint Commission as one that conditions justify? Give arguments to support your answer.

4. What recommendations are made concerning the teaching of arithmetic in the Second Report of the Commission on Post-War Plans?

5. What implications do you think those recommendations have for the teaching of arithmetic in the high school?

6. Give in fairly complete detail an outline of what you consider the arithmetical responsibility of the junior high school.

7. Do the same for the senior high school.

8. Do the same for the junior college.

9. What error is there in saying that $\frac{5}{15} = 0.33$?

10. Is 48.6 or 48.7 the closer approximation to $\frac{292}{6}$? How much closer?

11. Give the approximation to $\frac{2}{3}$ which has an error $\frac{1}{10}$ as large as the error in 0.667.

12. If three cans of food can be purchased for 35 cents, what is the exact price and what the approximate price of one can?

13. Which of the fractions $\frac{1}{8}$, $\frac{1}{7}$, and $\frac{5}{6}$ give rise to approximate numbers when expanded in decimal form?

14. How many significant digits in each of the following approximate numbers: 2.5; 2.05, 2.50; 250; 0.25; 2500, 250.0, 0.0025; 0.2500, 0.000002500, 0.0002050?

15. Which is the more precise measurement and which the more accurate in each of the following cases. 2.56 inches or 3,216 feet, 52.3 seconds or 15 seconds?

16. Determine the maximum apparent error, the relative error, and the per cent of error in each of the following measurements:

(*a*) 6.5 feet (*b*) 0.000020 inches (*c*) 5 inches (*d*) $5\frac{0}{4}$ inches (*e*) 0.005 centimeters (*f*) 117.200 miles per hour (*g*) $2\frac{1}{16}$ inches

17. What is the perimeter of each of these quadrilaterals:

(*a*) $6\frac{7}{8}$ inches, $5\frac{3}{8}$ inches, $12\frac{2}{8}$ inches, and $8\frac{0}{8}$ inches.

(*b*) 56.246 inches, 40.300 inches, 35.20 inches, 27.18 inches.

18. What is the area of the rectangle whose length is 16.72 inches and width is 8.46 inches?

19. Construct a unit on measurement which emphasizes how arithmetic and geometry may be correlated.

CHAPTER XII

THE TEACHING OF ALGEBRA IN THE JUNIOR HIGH SCHOOL

Algebra has long since acquired the widespread and unenviable reputation of being one of the most troublesome and difficult courses in the entire secondary-school program. It has acquired this reputation not so much because of any excessive difficulty in the subject matter itself as because of certain inherent *differences* between algebra and any study which the student will have encountered earlier in his educational career. A large share of the difficulties which students encounter in their study of algebra may be traced to the fact that algebra presents a radically new and different approach to the study of quantitative relationships, characterized by a new symbolism, new concepts, a new language, a much higher degree of generalization and abstraction than has been encountered previously; by the fact that, in contrast to arithmetic, algebra is more concerned with the conscious examination and study of processes than with particular answers to particular problems; and by the essential dissociation of many of its parts from intuition and concrete experience.

Too often, however, teachers either fail to recognize the essential significance of these characteristics of algebra, or else, through long-continued familiarity with them, permit them to become sheer habitual reactions. In either case the result is that teachers often fail to take account of the degree and the specific nature of the difficulties which these characteristics involve for beginning students. Unless teachers recognize these difficulties clearly and examine them with sufficient care to formulate specific ways for helping the students avoid or overcome them, the course is likely to degenerate into an aggregate of mechanical manipulations of symbols largely devoid of meaning. It is precisely this situation that has given rise to a large share of the current criticism of algebra as a school subject.

The fact that teachers often have been too preoccupied, negligent, or uninformed to take proper account of these difficulties does not mean that they are insuperable. On the contrary, experience has shown that through careful analysis and planning and skillful teaching a great deal can be done to obviate or minimize them. No single

topic is free of them. It is the job of the teacher to analyze each topic, to learn to anticipate the particular difficulties that are likely to occur in connection with it, and to plan to teach it in such a way that the difficulties may be avoided or forestalled as far as possible. Such a practice will go far toward enabling the teacher to explain away those difficulties which cannot be avoided altogether.

Algebra in the Seventh and Eighth Grades. The algebraic work to be included in the seventh- and eighth-grade courses should not be extensive and it should not be formalized to the extent that it will be in later courses. Its main objective should be to give the students an introduction to the meaning of certain useful and basic algebraic concepts such as literal numbers and formulas and the symbolic language of algebra. It is not intended that the algebraic work in these grades will lead to any large degree of technical skill in algebraic operations nor is it intended that this work shall be thought of solely in terms of its preparatory values. Rather it should be conceived as serving the double purpose of extending the conceptual background of the student as a sound transitional basis for later work and of providing mathematical experiences interesting in themselves and more general in nature than those encountered in the earlier arithmetic.

The influence of the suggestions made in 1923 by the National Committee on Mathematical Requirements gave rise to many new textbooks and courses of study in mathematics. Some of these, in their efforts to effect a thorough-going redistribution of material for the junior-high-school grades, attempted a more ambitious program of algebraic work for the seventh and eighth grades than now seems to be justified. Experience has indicated that probably it is better to confine the algebraic content of the work for these grades to a few fundamental, but simple, topics than to attempt too much.

The algebra that has been successfully introduced into grades 7 and 8 up to the present time has been limited largely to the understanding of the basic concepts, to the evaluation of formulas, and the solution of very simple equations. It seems possible and also desirable to include other algebraic material; but, if it is to prove effective, the work should be carefully planned and should be so organized as to be significant in itself as well as designed to furnish a good foundation for later algebraic study.[1]

[1] Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, *Fifteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1940), p. 80.

An earlier survey of a large number of courses of study indicated that the opinion expressed in this statement had a basis in fact and reflected accurately the prevalent practice in forward-looking schools.[1] It was revealed that the formula and the simple equation did occupy a far more prominent place in these grades than any other algebraic topics. However, it was not uncommon then nor has it been in more recent years for certain other topics to be suggested as being appropriate for these grades, among them being directed numbers, graphs, and simple verbal problems. In the hands of the skillful teacher these topics can undoubtedly be made meaningful and interesting to young students, provided that the treatment is kept at a level of difficulty commensurate with the experience and maturity of the students.

Although formulas, equations, etc., have been referred to as algebraic "topics," the reader should not get the impression that they are to be treated once, topically, and then relegated to the limbo of "finished work." On the contrary, they should be studied at recurring intervals and in various contexts throughout the course. Formulas and equations, in particular, are so pervasive and so amenable to gradation and adjustment from the standpoint of difficulty that appropriate applications of them can be made in connection with almost any topic which is likely to be considered in junior-high-school mathematics. Teachers should be alert to opportunities for making such applications and should become adept at using them to best advantage. At the same time, it is necessary to guard against the tendency to become overzealous with regard to this part of the course. In their enthusiasm teachers sometimes carry the algebraic work to a point of difficulty quite unwarranted in a seventh- or eighth-grade class. It should be kept in mind that the aim of the algebraic work in these grades is by no means an exhaustive coverage of algebra as such, but rather a good and progressively improving mastery of a few of its simpler concepts and processes as these are applied to familiar situations.

It is impossible to make a valid categorical statement as to what is the best arrangement and grade placement of this algebraic work in the seventh and eighth grades. The National Committee on Mathematical Requirements recognized the inadvisability of trying to make such a statement and presented instead five alternative plans. The weight of evidence, however, seems to indicate that, while certain work with simple formulas may be done satisfactorily in the seventh

[1] Edwin S. Lide, Instruction in Mathematics, *Bulletin* 17, Office of Education, 1932, National Survey of Secondary Education, *Monograph* 23 (Washington: Government Printing Office, 1933), p. 24.

grade, it is better to defer most of this algebraic work until the eighth grade. This is indicated in four of the five plans suggested by the National Committee and is in accord with practice as reflected in the majority of textbooks prepared in recent years for these grades. Moreover, it has the sanction of the Joint Commission[1] and the Commission on Post-War Plans.[2]

Such algebraic work as may be attempted in these grades should be informal, and it should be interesting. So far as possible it should be made to seem useful to the students. There should be little or no technical manipulation of symbols. The problem situations that are presented should be so simple that they can be readily associated with familiar arithmetical or geometrical situations so that by the process of analogy the appropriate procedures may be made to appear reasonable to the students. The main idea should be to give the students an understanding of the meaning of the language and symbolism of algebra as expressed in the formula and the simple equation and of the simplicity, power, generality, and importance of these mathematical tools.

Barber suggested excellent criteria for determining the appropriateness of algebraic material for the seventh and eighth grades and has given expression to the spirit in which algebraic work in these grades should be viewed:

> Any algebra which may be introduced into these grades should be subjected to three tests. Is it interesting? Is it useful? Is it thought-provoking? And to these there may be added a fourth: Does it prepare for the new algebra of grade nine? . . .
>
> In making his decision as to what algebra is to be used in these grades the teacher will note that the mensuration formulas are already included in what we call arithmetic. It is, at first, only a question of making the best educational use of such formulas in all the situations in which they are helpful.
>
> It is argued that, by the use of one-letter abbreviations and formulas, and, finally, equations, we can considerably improve the written work in a thoroughly sound manner; and that this provides a single method for handling many kinds of situations, and a method which becomes more and more useful and aids clear thinking as we go ahead with mathematics and its applications.
>
> The common use of the formula and the equation in the technical and the scientific world, even in its elementary and popular discussions, gives color to the claim that a working knowledge of these two tools of algebra

[1] Joint Commission, *op. cit.*, pp. 85–86.

[2] Commission on Post-War Plans, Second Report, *The Mathematics Teacher*, **38** (1945), 203–205.

has a general informational value which cannot be ignored in this day and generation. On the other hand to teach, in these grades, the old introduction to a mechanical algebra is worse than to teach no algebra at all.[1]

Most of the work in seventh- and eighth-grade algebra will be focused upon the reading of graphs, making clear the meaning of literal symbols, the use of these in setting up formulas and simple equations, and the evaluation of such formulas and the solution of such equations. Many commonplace relationships already familiar to the pupil give rise to formulas. Such relationships may be used to advantage by having the pupils translate their verbal expressions into symbolic language. For example, if pencils cost 5 cents each, the pupil readily states that the cost of 2 pencils will be 2 times 5 cents; the cost of 7 pencils will be 7 times 5 cents; and so on. He can be led without difficulty to generalize this situation to give the cost of any number of pencils at this price; *i.e.*, the total cost will be the number of pencils times 5 cents. It is an easy but important step for him now to pass from this verbal statement to the symbolic statement of the same relation: $C = n \cdot 5$ and eventually to the still more general statement $C = n \cdot p$. The evaluation of C for any given values of n and p thus becomes easy and meaningful.

Many familiar arithmetical problems can be made to yield just as satisfactory formulas and equations. Some examples are given in the following list, which could be extended easily.

In any uniform or average-rate motion, distance equals rate times time	$d = r \cdot t$
Simple interest equals principal times rate times time	$i = p \cdot r \cdot t$
Circumference of a circle equals pi times the diameter	$C = \pi \cdot D$
Perimeter of a rectangle equals 2 times the sum of the length and the width	$p = 2(l + w)$
The annual rent on a home equals 12 times the monthly rent	$R = 12 \cdot r$
Percentage equals base times rate	$p = b \cdot r$
Margin equals selling price minus cost	$m = s - c$
Area of a rectangle equals base times height	$A = b \cdot h$
Cost of sending a package by parcel post in the third zone is 13 cents for the first pound and 3 cents for each additional pound or fraction	$C = 0.13 + 0.03(n - 1)$
The diagonal of a square is equal to the length of one side times the square root of 2	$d = s \cdot \sqrt{2}$
The volume of a rectangular solid is equal to the product of the length, the width, and the height	$V = l \cdot w \cdot h$

[1] Harry C. Barber, "Teaching Junior High School Mathematics" (Boston: Houghton Mifflin Company, 1924), pp. 88–89.

The length of the hypotenuse of a right triangle is equal to the square root of the sum of the squares of the legs	$h = \sqrt{a^2 + b^2}$
The number of ounces is equal to 16 times the number of pounds..................................	$n = 16 \cdot p$
The number of gallons in a tank is (about) equal to 7½ times the number of cubic feet..................	$g = 7\frac{1}{2} \cdot f$
Number of centimeters is (about) 2.54 times the number of inches..	$c = 2.54 \cdot i$

The equations which are used should be so simple that the pupil will know intuitively how to solve them. So far as possible they, also, should grow out of familiar problem situations, although there can be no objection to setting up short lists of empirical equations for practice.

If John earned \$2.40 for 16 hours work, what was his average wage per hour?

This gives rise to the equation $16 \cdot w = \$2.40$, and the pupil's task is to solve for w. Intuition furnishes a sufficient guide for this.

Fred and Joe picked 40 quarts of cherries one day, but Joe picked 8 quarts more than Fred did. How many quarts did each pick?

This suggests the equation: $F + (F + 8) = 40$, or $2F + 8 = 40$. By subtracting 8 from each side of the equation it becomes $2F = 32$, for which, again, the solution is effected intuitively.

Such equations provide an easy and natural approach to the more formal methods which will be used later. They need not and should not be made difficult. Speed of solution is not a consideration here. The main considerations at this stage should be to ensure understanding of the derivation of the equation, to ensure understanding of the solution, and to develop the habit of using such equations in solving problems.

Ninth-grade Algebra. In the ninth grade more teachers and more students spend more time and effort in a more extensive study of formal algebra than in any other grade in the entire secondary-school system. In the combined seventh and eighth grades of the junior high school the total number of students having some contact with algebra is perhaps greater than in the ninth grade, but the algebra treated in these grades is extremely limited in scope and informal in character. On the other hand, the more extensive and formal algebra of the senior high school and the junior college affects only a relatively small group of students who are more highly selected. It is in the ninth grade that the serious study of the subject begins for most students, and it is with this grade that it ends for many of them. Here the student's interest is either kindled and nourished or allowed to die. Here he

either gains those apprehensions and skills which are necessary for further progress, or else he fails to gain them and so has his way blocked to the pursuit of further study in this field or to extensive study in many related fields. All the concepts, principles, and procedures of ninth-grade algebra carry over into the work of later years and in fact form the very foundation of that work. Thus the ninth grade is the most critical grade so far as algebra is concerned. Here lies the teacher's greatest challenge and his greatest opportunity.

Literal Numbers and Formulas. The formula, a symbolic statement of relationship between two or more variables, provides an ideal medium for the transition from the earlier work to the more formal and systematic aspects of algebra, and a theme about which a great deal of the work of the ninth grade can be organized. It involves or is closely associated with a great many of the concepts of elementary algebra; the symbolic language of literal numbers, constants, and variables; the concept of dependence and function; graphic representation of relationships; substitution and evaluation; and operations with signed numbers, literal numbers, parentheses, exponents, fractions, radicals, etc. Thus it forms a core which has points of contact not only with the previous experiences of the students but with many of the topics which will be considered subsequently throughout the ninth-grade course and in later courses.

The ninth-grade student is not entirely unacquainted with formulas even when he begins his study of algebra. In his previous work in arithmetic and informal geometry he will doubtless have had some contact with such formulas as those for simple interest and for the mensuration of the simplest and most common geometric forms. If he has had a thoroughly good course in seventh- and eighth-grade mathematics, he should have attained some understanding of the significance and generality of the formula as a shorthand statement of relationships between quantities and as a rule for operation. He will have had some experience in evaluating simple formulas and perhaps will have constructed or set up a few simple formulas from quantitative situations within his experience. He thus brings to his ninth-grade work enough background to enable him to use the formula as a point of departure in his work in algebra, and the further study of the formula, in turn, serves to familiarize the student with the new language, concepts, symbolism, and operations of algebra and to give him experience in the progressive mastery of these.

The main things which the student should get from his study of formulas in the ninth grade are these:

1. An understanding and appreciation of the nature and significance of literal numbers and of the symbolism of algebra
2. A clear concept of the meaning of a constant, of a variable, and of the distinction between the two
3. A clear concept and appreciation of dependence and of the meaning and relationship of independent and dependent variables
4. The ability to set up simple formulas expressing relationships existing in situations within the student's experience
5. Facility and accuracy in substitution in and evaluation of formulas
6. The ability to represent graphically the relationships indicated by formulas involving two variables
7. The ability to solve formulas, *i.e.*, to transform an implicit relationship into an explicit relationship through application of the laws of algebraic operation

If the teacher will organize the work relating to formulas around these main foci and will consciously plan every exercise and activity so that it will bear upon and contribute to the attainment of one or another of these main objectives, then worth-while results may be expected. Otherwise the work is likely to follow a too prevalent piecemeal pattern whose only plan is the order of topics and exercises in the textbook; a pattern in which organization, emphasis, and direction will be lacking and from which any worth-while outcomes which may occur must be regarded more as welcome accidents rather than as legitimate expectations.

One of the first of the fundamental tasks which the student faces in the study of formulas and of algebra in general is to acquire a good understanding of the real meaning of *literal* numbers. It is customary to introduce students to the meaning of literal numbers by having them consider common situations in which the relationships between two or more elements are known; to have them state these relationships in words and then abbreviate the verbal statements by substituting letters and symbols of operation and equality for the words. For example, the student may be asked to tell how to find the distance which an automobile traveling at a uniform speed will cover in a given time. His statement will probably be to the effect that "the distance is equal to the rate of speed (in miles per hour) multiplied by the time (expressed in hours)." He can be led easily to see that, by using letters to represent the verbal expressions, he can write this same relationship more briefly and conveniently as $d = r \cdot t$. That is, in this simple case he has little difficulty in associating the letter d with the meaning "distance," r with the meaning "rate of speed," and t

with the meaning "time in hours." He thus gains almost at once the important idea that a letter may stand for a meaning which can be expressed more elaborately in words.

This, however, is not the whole story. Unless the student is made keenly aware, not only that the letter is to be associated with a verbal expression (*e.g.*, *r* for "rate"), but that it must be identified in any particular instance *with a number*, he is likely to come to the erroneous and meaningless conclusion that "miles per hour times hours equal miles" or that "feet times feet equals square feet." Such statements are not at all uncommon, but they exhibit a lack of clarity with respect to the fundamental meaning of literal numbers. The student must be made to understand that literal numbers represent primarily and essentially *numbers* although they may refer to the enumeration or numerical measurement of some particular kinds of objects or magnitudes.

This concept is fundamental to the clear and precise understanding of the nature of literal numbers and of formulas. The use of literal numbers can probably be developed most effectively through numerous illustrations of the use of letters to represent such things as lengths of line segments, weights, sizes of angles, or unknown quantities in simple verbal problems or equations. Such illustrations should be closely associated with repeated and closely supervised practice in the actual evaluation of formulas by the direct substitution of specified numbers and the performance of the indicated operations after the substitutions have been made. The teacher should employ illustrations of these procedures freely, because the immature mind responds much more readily to illustration than it does to definition or verbal direction. During this period of development and early practice in the employment of literal numbers, the work of the student should be under the close supervision of the teacher in order that any misconceptions and mistakes may be detected and corrected at the outset and so be prevented from becoming fixed habits.

Literal numbers are often used to represent *variables*. The two are not synonymous because sometimes literal numbers may represent constant quantities or numbers, but in our conventional symbolism, variables are always represented by letters. The concept of a variable is indispensable to the full understanding of the nature of dependence and of formulas. The ordinary experiences of children furnish innumerable illustrations of both variable and constant quantities. The fact that many children come through one or more years of work in algebra without any clear understanding of what is meant by the term "variable" can mean only that teachers do not take the trouble to

present a sufficient number or variety of these illustrations and to emphasize specifically the characteristics of variation. Changes in age, height, or weight of individuals; the distance of a moving body from a fixed point; changes in temperature; etc., are but examples of many familiar situations which could be used to make clear the meaning of variation and of variable quantities. If these illustrations are to yield the desired concepts, however, the teacher must see to it that the attention of the students is focused upon the characteristics of variation, and this emphasis must be made repeatedly and specifically. One cannot legitimately expect any clear concept of a variable to emerge as a mere incidental by-product. It is not difficult to teach, but it must be taught and taught specifically if it is to be really mastered by any substantial majority of the students.

The concept of *function* or *dependence* is so important that it has been called the "unifying element of all mathematics," and a great deal has been written and said about it. But, in spite of this, relatively few students ever come to have a real appreciation of its significance or a thorough understanding of its nature. Indeed it seems probable that teachers themselves generally fail to be aware of, and sensitive to, this omnipresent concept of dependence. At least they miss innumerable opportunities to give their students a fuller understanding of the function concept and to keep them conscious of functional relations and alert to recognize situations which involve dependence among quantities. The recognition of such dependence and the determination of its nature is essentially what constitutes functional thinking.

The function concept, or the idea of dependence of one element in a situation upon one or more other elements, is not inherently difficult to develop in children. The reason why it is usually so inadequately developed is because specific attention is so seldom given to it. It seems to be one of those outcomes which many teachers wishfully hope for as a by-product of instruction but which they erroneously regard as so sure of occurrence that to give direct and specific attention to it would be a waste of time.

As is the case with most elementary concepts, the best method of development lies in illustration. The concept of dependence is best illustrated by taking cases involving related quantities such as the diameter and circumference of a circle and showing that, when one of these is known, the other is uniquely determined, and that a change in either of them will produce a corresponding change in the other. In this connection it is helpful to have the students build tables by com-

puting and tabulating the values of the dependent variable (or function) which correspond to arbitrarily assigned values of the independent variable or variables and then to make graphs from these tables of variables, as illustrated in the accompanying example. It is possible, of course, for students to perform the necessary substitutions and computations in a mechanical way and without *any conscious recognition* of the interdependence of the quantities involved. It is largely because teachers have failed to stress this conscious recognition of dependence and to keep it continually in the focus of attention of the students that the notion of functionality has not played the

TABLE AND GRAPH SHOWING RELATION OF LENGTH OF CIRCUMFERENCE OF A CIRCLE TO LENGTH OF DIAMETER

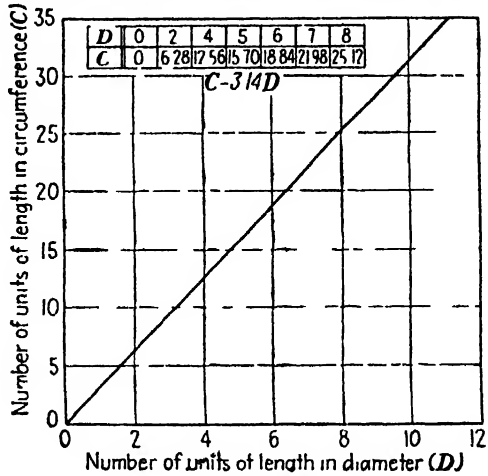

Fig. 6.

important part which it should in mathematical education. It is not that it is hard to understand, for it is not. If the concepts of dependence and function are to be made to play an integral and basic part in the mathematical thinking of students, teachers must continually, deliberately, and specifically bring them to the attention of the student and keep his attention centered upon them by repeated illustrations of their occurrence.

The concepts of *dependent* and *independent variables* are implied in the understanding of dependence. But here again the differential characteristics of these concepts are not likely to emerge and stand out with distinctness in the minds of the students unless the teacher directs special attention to them and shows by repeated illustration just *why* they are called variables and why one of them is designated as an *independent* variable while the other is called a *dependent* variable.

Attention has already been called to the practice of having students *translate verbal statements of relationships into formulas* by the use of literal numbers to represent the related elements in the situation. There can be little doubt that such practice in translating statements or laws into formulas is of great value in centering the students' attention upon the fundamental concepts which have been discussed and in clarifying these concepts. Probably the simplest and most effective way to give students an understanding start in this use of symbolism is to show them that it may be regarded as a sort of shorthand method of writing down what would otherwise have to be written in a less convenient verbal form. The contrast between the verbal and the symbolic forms and the advantage of the latter can be emphasized by actually writing down the verbal statements of relationships and then "for the sake of convenience" rewriting the statements by using merely the initial letters of the key words rather than the words themselves. The following examples illustrate this method.

Distance equals rate of speed multiplied by time

$$d \quad = \quad r \quad \times \quad t$$

Cost of gasoline equals number of gallons multiplied by price per gallon

$$C \quad = \quad g \quad \times \quad p$$

A few such examples will serve to enable most students to get the idea of what is being done and to learn to appreciate the significance of the symbolism and of the expressed relationships. However, in order to ensure something distinctly beyond a mere threshold understanding of these concepts, the students should be given a substantial amount of practice in this work, and the practice should be spread over a considerable period of time.

Teaching Students to Solve Linear Equations. One of the most common and important activities of ninth-grade algebra is the solution of linear equations. Presumably the student will bring to his study of algebra some understanding of what an equation means, since he will have had experience with simple equations in his previous work in arithmetic and informal geometry and perhaps even with certain very simple equations in which the unknown quantity is represented by a literal symbol. In all probability, however, his experience in solving equations will have been hardly above the intuitive level. For example, if confronted with the equation $3n = 24$ and required to find the number represented by n, he will probably reason that, if

three n's make 24, then "it stands to reason" that one n will have to be one-third of 24, or 8.

These intuitive reactions are generally sufficient and satisfactory so long as the situation is very simple, *i.e.*, so long as it is possible for the student to keep in mind clearly and simultaneously all the pertinent elements and relationships which are involved in the situation. On the other hand, the moment a problem situation becomes so involved that he is unable to keep all the elements and their proper relationships clearly in mind at the same time, intuition breaks down, and when this happens logic must take its place. In such cases the only recourse is to more formal and powerful tools for the analysis of problem situations. Such a tool is the algebraic equation.

Since the turn of the century much criticism has been directed against formalism in secondary-school algebra. It is undoubtedly true that the mechanical aspects of algebra have been heavily stressed and that the emphasis placed upon the formal operations has too often given little or no consideration to underlying meanings. There is a great difference, however, between formalism, conceived in this sense, and the formalization of mathematical procedures. Algebra really consists fundamentally in the generalization and formalization of these procedures, but this formalization need not and should not be divorced from meanings. Rather it should be conceived as merely an extension of familiar procedures into an environment of number concepts more general and more powerful than those of elementary arithmetic. The student should be taught to look upon the algebraic equation as a device which enables him easily to investigate relationships which would be too complex to be investigated successfully or easily without its aid. It should be explained to him that the solution of formulas or equations operates under certain fixed laws called "axioms." He should learn the meaning of these axioms. They should be explained to him and illustrated by the teacher in terms of the familiar quantitative concepts of his past experience. After he has thus been given a feeling of the reasonableness of the axioms, he should be given practice in using them not only with arithmetical numbers but with literal numbers as well.

The fundamental operational axioms involved in the solution of linear equations are as follows:

1. If equal quantities are added to equal quantities, the results are equal.
2. If equal quantities are subtracted from equal quantities, the results are equal.

3. If equal quantities are multiplied by equal quantities, the results are equal.

4. If equal quantities are divided by equal quantities, other than zero, the results are equal.

The student should learn to react without hesitancy to these axioms. They should become so much a part of him that he will come to apply them as readily to literal numbers as to ordinary arithmetical numbers in an equation.

Linear equations in ninth-grade algebra assume a variety of forms, illustrative of which are such forms as:

$$\begin{array}{lll} ax = b & x + a = b & ax + b = c \\ \frac{x}{a} = b & a - x = b & x - a = b \\ & ax + b = cx + d & ax + bx = c \end{array}$$

While these forms are all variations of a common form, the similarity is usually not immediately apparent to children encountering them for the first time. Moreover, textbooks and teachers are frequently deficient in giving emphasis to this point. As a consequence the different forms are often taught separately, a special technique being developed for each, much as "the three cases" of percentage are often taught in arithmetic. Such a practice probably makes for quick mastery of the separate techniques but for little else. It is in fact precisely this type of treatment against which the legitimate criticism of algebra has been directed. It does not and cannot make other than an incidental contribution to the development of any real power of generalized understanding and original analysis. It produces little more than the acquisition of certain transitory skills, and so it largely fails in the attainment of its real objective.

A much sounder procedure would be to try to find a single unifying principle which would be applicable to all forms. Such a principle fortunately is available and can be stated in a way which is clearly understandable to ninth-grade students. It can be formulated in three or four key sentences somewhat as follows:

1. In solving *any* linear equation in one unknown for the unknown (we may call it x) the object is to get an equation in which x will stand alone on one side of the equation and will not appear on the other side.

2. In order to do this we must get rid of all the other numbers or letters which are associated with x on its side of the equation.

3. We get rid of these numbers or letters by undoing the operations which associate them with x; that is, by applying the processes which are the inverse (opposite) of those which bind these letters or numbers to x.

4. If any operation is performed on one side of an equation in order to change its value, the same operation must be performed on the other side of the equation, because if it is not we shall no longer have an equation.[1]

This principle gives a basis for the solution of linear equations without any recourse whatever to intuition, except the intuitive feeling that the only way to make a number symbol stand alone is to get rid of the other number symbols that are connected with it. It gives emphasis to the character of the equation and lends organization and generality to the solution of linear equations which, in turn, eliminate any necessity for developing special methods for the different forms. For the student who follows this general plan, the specialized procedures will emerge automatically as he finds need for them since they are but adaptations of the general plan to particularized forms of the equation.

For example, let it be required to solve $ax - b = c$ for x. In order to make x stand alone on one side of the equation, it is necessary to get rid of the b and the a from that side of the equation. Since the x appears in only one term, we may first get rid of the other term, b. This is done by adding b to both sides of the equation, since originally b was subtracted from ax and any subtraction can be undone only by the inverse process, addition. This[2] gives $ax = c + b$. It is now necessary to get rid of the a from the left member of the equation. Since the a is a multiplier of x, we can get rid of it only by undoing the multiplication. For this purpose we must employ the inverse process, division. That is, we must divide both sides of the equation by a. This will give the required solution: $x = (c + b)/a$.

Sometimes the unknown appears in more than one term. When that occurs, it is necessary first to collect these terms and to factor out the unknown as a common factor. After this has been done, the solution is identical in nature to the one that has just been described. For example, let it be required to solve

$$ax + b + cx = d \text{ for } x.$$

(1) Collect the terms in x	(1) $(ax + cx) + b = d$
(2) Factor out x as a common factor	(2) $x(a + c) + b = d$
(3) Get rid of the term b from the left member of the equation by subtracting b from both members	(3) $x(a + c) = d - b$

[1] Paul Ligda, "The Teaching of Elementary Algebra" (Boston: Houghton Mifflin Company, 1925), pp. 48–52.

[2] This is the equivalent of "transposition," which may be used if the students are accustomed to it.

(4) Get rid of the multiplier $(a + c)$ from the left member by dividing both members by $(a + c)$

(4) $\frac{x(a+c)}{(a+c)} = \frac{d-b}{(a+c)}$

or

$x = \frac{d-b}{a+c}$

This gives the desired solution.

It is hardly necessary to give further illustrations. The single unifying principle of "getting rid" of the unwanted terms, divisors, and multipliers by the application of inverse processes results automatically in the solution of all forms of the linear equation in one unknown. Approached intuitively in the beginning, its complete reasonableness can be made apparent to children without difficulty. Thereafter they should be led to focus their attention on the process per se and to give less and less attention to its concrete numerical setting. Thus eventually it will stand out as a general, abstract, mechanical principle of operation, not devoid of meaning because it will have been built upon meanings in the beginning, but no longer dependent upon intuition, and therefore more certain and more powerful than the earlier and less formal procedures

The Evaluation and Solution of Formulas. Theoretically the *evaluation of formulas* presents no learning difficulties. Actually, however, students make mistakes in this simple process, and they are not always mistakes in computation. Mistakes in substitution occur with unexpected frequency. These are most often associated with the rewriting or recopying of the formulas with the letters replaced by corresponding numerical values. This type of error can be offset to a considerable degree by having the students make a practice of enclosing in a separate parenthesis each numerical value which is substituted for a letter. This has a tendency to focus attention upon each quantity as a separate element in the formula and to avoid the confusion of one such element with another. In blackboard work it is helpful to have the students actually erase one by one the letters in the formula and to write in the numerical value of each letter as that letter is erased. This makes the students keenly conscious of the fact that *the letters are actually to be replaced by the numbers*, and thus strengthens the appreciation of the real meaning of evaluation. While this practice of erasing and rewriting *in situ* cannot be followed so satisfactorily where the work is being done with pencil and paper, something of the same effect can be attained by having each substituted number enclosed in a separate parenthesis and written in a place which precisely corresponds to the place occupied in the formula by the letter for which the substitution is being made.

The evaluation of formulas involving only two variables can also be tied up with the graphs representing the relationships. Assuming that the graph has been constructed, any value of the independent variable (lying within the range covered by the graph) may be selected. When this value is referred to the corresponding point on the graph and that point, in turn, to the axis of the dependent variable or function, the value of the latter is given (at least approximately) at once. To illustrate, use the relation for the number of pounds to the number of kilograms: $P = 2.2K$. If we let K have the value 2.5 we immedi-

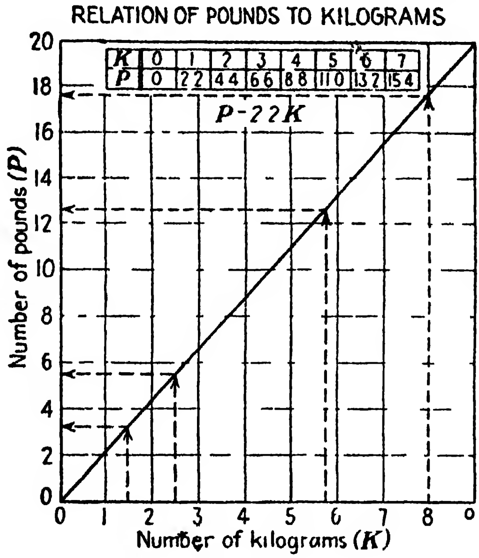

FIG. 7.

ately determine P as being (at least approximately) 5.5. Similarly, if K is $5\frac{3}{4}$, the corresponding value of P is found to be about 12.7; for $K = 8$, we get $P = 17.6$; etc.

The *solution of formulas* often causes students much difficulty. This is because the students are not made consciously and specifically aware of the general principles underlying the solutions. Indeed, teachers themselves often seem to be unaware of these general principles. The principles do exist, however, and are quite simple and capable of being applied in an understanding manner by ninth-grade students. They are merely the principles which underlie the solution of all simple equations, whether numerical or literal, integral or fractional, rational or irrational.

The general procedure may be illustrated by considering the formula $A = P + Prt$. Let it be required to solve for t in terms of A, P, and r.

Since the letter t occurs in only one term, that term must be retained and all other terms eliminated from that member of the equation. This elimination of elements from one member of the equation (or formula) is always accomplished by *undoing* the operation which binds that element to the rest of the given member of the equation. In this case P is related to Prt by addition. Hence, to eliminate P from that member of the equation we must undo this addition, or, in other words, we must subtract P from the right member of the equation. It, therefore, becomes necessary to subtract P also from the left member of the equation. This (the equivalent of transposing P) gives the equation $A - P = Prt$.

Now since we wish to solve for t we must eliminate Pr from the right member. But Pr is bound to t by multiplication. Hence, in order to eliminate Pr and make t stand alone, we must undo the multiplication by using a process which is the inverse of multiplication, *viz* , division. Therefore, we divide the member Prt by Pr, and, of course, if we divide one member of an equation by a given quantity, we must also divide the other member by that same quantity. Thus we get $\frac{A - p}{Pr} = \frac{(Pr)t}{(Pr)}$ or $\frac{A - p}{Pr} = t$. This principle of "elimination by undoing" (or by applying opposite processes) is perfectly general and is not difficult once the students are brought to see it in its essential simplicity. It removes the solution of formulas from the status of a bag of tricks and places it upon a reasonable basis.

Apparent complications are introduced when the student is required to solve for a letter that occurs in two or more terms. In such cases, however, it should be pointed out that it is merely necessary first to group these terms and take out the required letter as a common factor. The procedure then follows the general pattern indicated above. To illustrate: let it be required to solve the foregoing formula for P in terms of A, r, and t. The steps would be as follows:

$$A = P + Prt$$
$$A = P(1 + rt)$$
$$\frac{A}{(1 + rt)} = \frac{P(1 + rt)}{(1 + rt)}$$
$$\frac{A}{(1 + rt)} = P$$

Note that, after the right member has been factored, it remains merely to eliminate the factor $(1 + rt)$ by undoing the multiplication, *i.e.*, by dividing both members of the equation by that factor.

The Teaching of Graphs. In recent years an increasing amount of importance has been attached to the study of graphs in junior-high-school mathematics. Among the several reasons for this may be mentioned the interesting character and the practical importance of graphical devices; the simplicity and power of the graph for presenting data in a condensed, understandable, and striking way; and the increasing prominence of graphical devices in newspapers, magazines, and other current publications.

There are two fundamentally distinct types of graphs, *viz.*, the statistical graph, and the mathematical or functional graph. The former is a device used to picture the relationship that exists between several different quantities which are comparable but are not necessarily interdependent, while the latter is used to picture the relationship that exists between two or more variables whose values are so related that they are dependent on each other. Of the statistical graph there are, according to Karsten, four distinct types which may be classified as abstract, geographical, frequency, and historical. The nature of the graph or chart is, of course, a function of the distribution of the data to be represented.[1] Among the more frequently used bases for classifying the functional graph there are (1) type of relation, *e.g.*, linear, quadratic, etc.; (2) type of curve, *e.g.*, straight line, circle, parabola, ellipse, hyperbola, sine, tangent, etc.; (3) continuity; and (4) multiplicity of values.

From either point of view the graph is an effective means of presenting data, making comparisons, and depicting relations; it offers untold opportunities for free play of the imagination, for the application of simple or ingenious constructive abilities, and for the development of an enthusiastic interest in mathematical methods and a more intelligent understanding of fundamental procedures on the part of all those becoming proficient in its construction and interpretation. As the minimum contribution it should make to the program of attaining this proficiency, the secondary school should develop the ability (1) to construct and interpret bar, broken-line, curved-line, and circle graphs in the presentation of statistical data; (2) to make comparisons between various comparable statistical graphs; (3) to recognize the characteristics of data to be represented by each of these four types of statistical graphs, as well as certain fundamental cautions that are to be observed in their construction and interpretation; (4) to construct and interpret a functional graph as referred to a reference frame of coordinates;

[1] Karl G. Karsten, "Charts and Graphs" (New York: Prentice-Hall, Inc., 1925), p. 675.

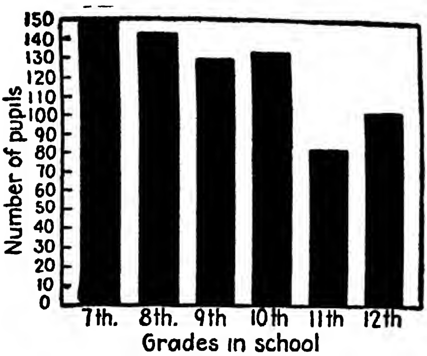

Fig. 8. Number of pupils enrolled in a city high school

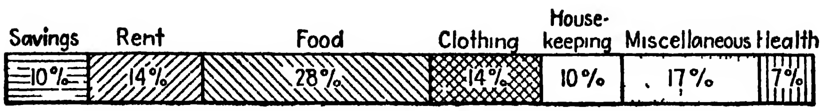

Fig. 9. One hundred per cent bar graph of a family budget.

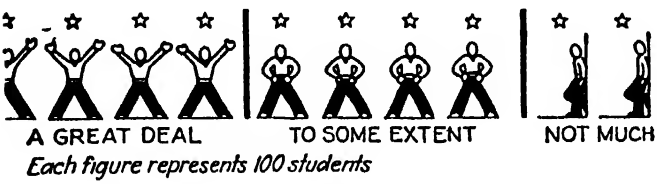

Fig. 10. Do students in a certain school like the school?

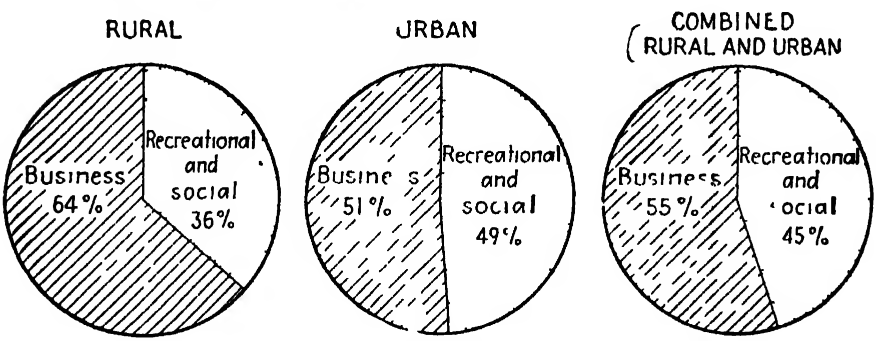

Fig. 11. Passenger automobile mileage used on business

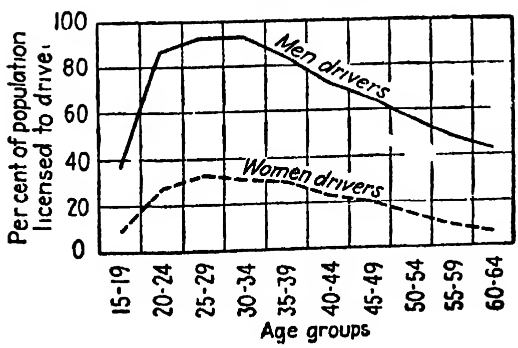

Fig. 12 Men and women licensed to drive.

(5) to use the functional graph in solving algebraic equations and to understand the simpler geometric implications; (6) to interpret the graph in the light of the functional dependence shown, including simple maximum and minimum values.

Frequently the assumption is made that the complete understanding of graphs is implied and guaranteed by the ability to construct them. Such an assumption, however, is unwarranted. It is entirely possible for a student to plot a series of points whose coordinates satisfy a particular equation, to draw a smooth line through these points, and to call this the "graph of the equation" without having any clear

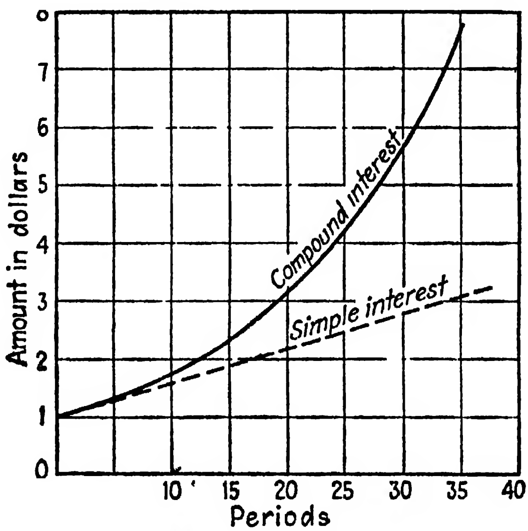

FIG. 13. Amount of $1 at 6 per cent compounded annually.

realization of the meaning of what he has done. This is not to say that the construction of graphs is unimportant. On the contrary, not only is it extremely important in the development of a full understanding of the meaning of graphs, but it gives valuable review and practice in understanding the meaning of the coordinates of a point and in the solution of equations and the substitution of numbers leading to the evaluation of algebraic expressions. The construction of graphs and the study of their meaning should go hand in hand.

The customary way of teaching students how to make graphs is to give them an equation such as $2x + 5y = 9$ and to have them build a table of number pairs which satisfy this equation. They are then shown how to locate points by means of these number pairs. After a few points have been located, the students are instructed to draw a

smooth line through them. Too often the subject is carried no further than this. Frequently teachers fail to give adequate instruction even with reference to such fundamental matters as the dependence of one of the variables upon the other, the arbitrary assignment of values to the independent variables, the naming of the axes, and the selection of suitable scales.

The construction of graphs can contribute little toward the development of adequate concepts of variation, continuity, and dependence if it is taught in this purely perfunctory fashion. These concepts are not likely to enter into the students' thinking unless they are specifically pointed out, not once but many times, by the teacher. Many and varied illustrations should be used. The students will find within their own common experiences many situations involving relationships among variable quantities that may be appropriately subjected to graphical treatment and which, because of their familiarity, will help materially in giving meaning to the graphs.

As an illustration, consider the relationship between the amount of gasoline that goes through a pump at a filling station and the total cost of this gasoline at 26 cents a gallon. This provides an excellent situation for emphasizing the *dependence* of the one variable, the cost C, upon another variable, the number of gallons N, the price remaining constant. It can be expressed by the formula $C = 0.26N$. This formula can then be used as a basis for constructing the graph (Fig. 14) After the graph has been made, it should be carefully re-examined with attention directed to the way in which it answers such questions as: What happens to the cost as the number of gallons increases? As the cost increases, does the number of gallons increase in the same ratio? Does a decrease in either of the variables bring about a corresponding decrease in the other? How does the price per gallon affect the direction of the graph? If the price were increased, how would the direction of the graph be changed? Would the selection of different scales for numbering the axes cause the direction of the graph to be different from its present direction? Approximately how many gallons of gasoline could be bought for \$1? For \$1.50? For \$2? For \$3? Does the graph show the cost of 8 gallons? For *any* point on the graph should the ordinate C give a number exactly 0.26 times as great as the number given by the abscissa N of that point? Would the same result be given by the formula? If a point were to move along the graph, would its ordinate or its abscissa change the more rapidly? How many times as rapidly? Explain how the graph shows that the cost *depends* on the amount of gasoline bought. Explain how

the graph shows that the amount one could buy *depends* upon how much money he could spend for gasoline.

By such questions as the foregoing the students can be made conscious of the graph as a device which shows in a striking way the fact and the precise nature of the dependence of either of the variables upon the other, of the meaning of dependent and independent variables, and of the precise way in which a change in either variable inevitably brings about a corresponding change in the other variable. The students

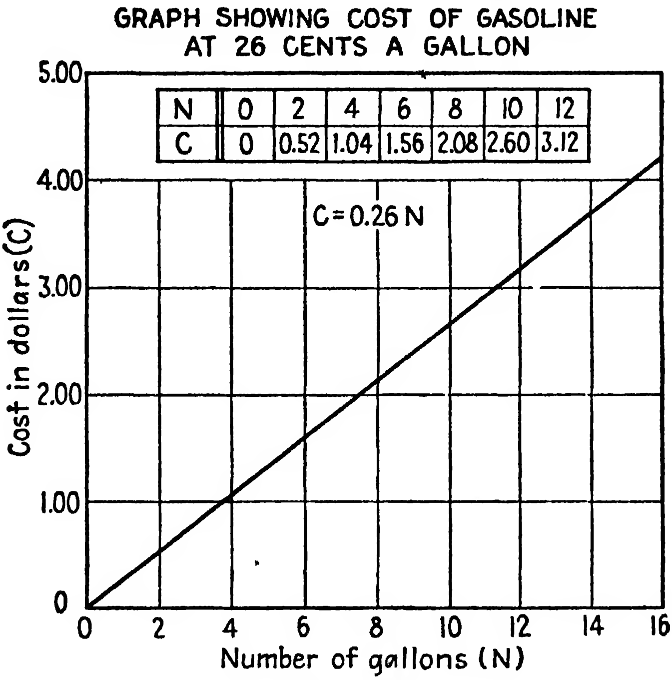

FIG. 14.

become more clearly aware of the meaning of coordinates. They learn to associate the relationships shown by the graph with those indicated by the formula or equation upon which the graph is based. They learn, in short, to understand what a graph means, and this, in turn, enhances their apprehension of dependence and functional relationship as a permeating principle, whether it is expressed graphically or by means of formulas or equations.

An associated problem calling for the graphs of two simultaneous equations could be set up as follows: "One filling station sells gasoline at 26 cents a gallon. A competitor advertises '20 cents a gallon plus 30 cents service charge.' Make cost graphs for both stations on the

same set of axes." This problem involves setting up a second formula, $C = 0.20N + 0.30$, and the construction of another graph along with the one already made and discussed. Questions similar to those heretofore indicated should now be discussed with reference to the new graph (Fig. 15).

In addition, such questions as the following should be discussed: With what amount of money could one purchase the same amount of gasoline at the second station as at the first? At which station could one get the most gasoline for \$1? For 50 cents? For \$2? How much more gasoline could be purchased at one station for \$2.50 than

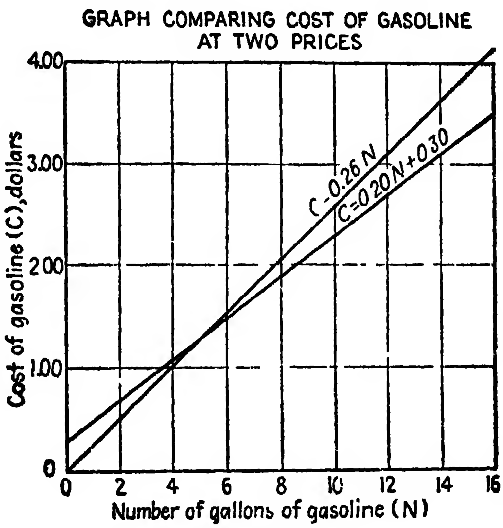

FIG 15.

at the other station? At which station could one buy 10 gallons at the lower cost? At which station could one buy 2 gallons at the lower cost? How much could one save buying 15 gallons at the station that offered the lower price on this amount? Show how to find this out from the graphs. Does the service charge made by the second station affect the direction of the graph? Does it affect its position? Explain. Does the price per gallon affect the direction of the graph? Explain. Would a change in the direction of the graph indicate a change in the price per gallon? Explain.

Consideration of the direction of a graph leads naturally and easily to the concept of *slope* which is also to be associated with the rates of change of the two variables and with the coefficient of the independent variable in the function, formula, or equation. Similarly, the point

where the graph crosses the axis of the dependent variable should be associated with the constant term in the function, and the effect which any change in this constant term has upon the position of the graph should be studied. The students will be interested in noting that, when the equation is written (rewritten if necessary) in the form $y = mx + b$ (or $C = 0.20N + 0.30$), the b always indicates the point at which the graph crosses the axis of the dependent variable and that the m always indicates the slope. Thus, by solving any linear equation of the form $ax + by = c$ for y, the student has at his command a method for constructing the graph which is at once less tedious and more meaningful than the method described earlier in this section.

Attention should be called to the fact that the m and b, or the *slope* and the *y-intercept*, are two conditions that determine the position of a straight line just as two points determine its position. The method of determining the x-intercept and y-intercept from the equation of any line and their use in plotting the graph of the line should then be emphasized. All this discussion of the linear equation and its graph should lead to the summarizing generalization: To determine the position of a straight line in a plane, it is necessary to have two independent conditions.

Specific attention to these considerations not only adds interest and value to the study of graphs but, in an easy, natural, and understandable way, provides the beginnings of a sound technical foundation for a real understanding of later work in analytic geometry and calculus. Obviously more time is required for this sort of treatment of function graphs than would be required for the mere construction of the graphs themselves. One may feel sure, however, that the thorough discussion of a few instances along the lines which have been indicated will do more to give the students a sense of the functional relationships involved than will the mere rule-of-thumb construction of large numbers of graphs.

The Teaching of Directed Numbers. The study of directed numbers is an integral part of the study of algebra. It is also one of the most difficult topics to develop successfully, and teachers and writers are not altogether agreed upon the most satisfactory methods of teaching it. There is general agreement, however, as to the main outcomes which are desired. These may be clearly stated as follows:

1. The student should gain an understanding of the meaning of directed numbers.
2. He should be led to see that the operations with directed numbers are consistent with the operations of arithmetic and that they constitute a more

generalized procedure in which the operations of arithmetic appear as special cases.

3. He should gain considerable facility in performing the fundamental operations with directed numbers.

The fact that students fail to attain adequate mastery of these objectives undoubtedly accounts for a great deal of the difficulty which they experience in the study of algebra.

The student's numerical experience prior to the introduction of the concept of negative numbers will have been confined entirely to dealings with the numbers of arithmetic, *i.e.*, with numbers representing quantities actually greater than zero. Up to this time he has used *zero* in two capacities, either as a number or a placeholder in writing numbers such as 305 and 500. Now, however, it becomes necessary to give to zero a new significance. In addition to its use as a number and as a placeholder, *zero* will now be regarded as an arbitrary starting point in the number scale from which one may count in either direction; numbers counted in one direction will be called "positive" numbers while numbers counted in the opposite direction will be called "negative" numbers. Many illustrations of this new use should be given.

The number scale is probably the most satisfactory and helpful of all devices for making clear the nature of positive and negative numbers and for illustrating their characteristics of oppositeness, direction, and position. It should be used, however, in connection with other devices for illustrating the opposite character of positive and negative numbers and the arbitrary selection of the zero or reference point. Illustrations of assets and liabilities, north and south latitude, temperatures above and below zero, etc., are helpful in developing the fundamental concepts of oppositeness, direction, position with reference to an *established* zero, but they present only a partial picture of the nature of positive and negative numbers. They fail to make clear that whether a number is to be regarded as positive, negative, or zero relative to some other number depends not only upon its own position in the scale but also upon the position of the number to which it is referred, and that that number may or may not be the previously established zero. Thus we may speak at 3:00 P.M. of an event which happened, say, at 1:00 P.M. Its position in time, using 1 hour as a unit, would be indicated by -2 if reckoned from *now* (three o'clock) but would be indicated by $+1$ if reckoned from noon, or by $+13$ if reckoned from the previous midnight (Fig. 16).

Referring this situation to the number scale, it is seen that the event's position in time is positive with reference to *any number* that

lies to its left on the scale and is negative with reference to *any number* which lies to its right on the scale. Thus the statement "all negative numbers are less than zero and all positive numbers are greater than zero" is true only when the pre-established zero is the number to which all other numbers are referred.

The difficulties incident to the mastery of the concept of directed numbers render it an unsuitable topic with which to begin the study of algebra. In years past, numerous textbooks have introduced directed numbers at the very beginning of the course, but experience has shown that it is better to postpone this until the students have become thoroughly familiar with some of the other new and basic concepts, particularly those of literal numbers and formulas.

There is considerable difference of opinion among writers and teachers as to the lengths to which teachers should go in the effort to

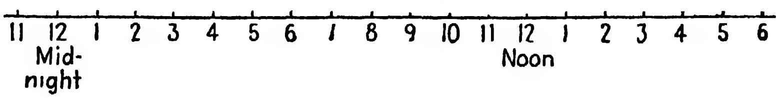

Fig. 16.

rationalize the operations with directed numbers, and to explain them in terms of the familiar operations of arithmetic. It is undoubtedly desirable to have these operations explained in such a way that they will be manifestly consistent with established arithmetical operations. On the other hand, some of the attempts which have been made to rationalize operations with directed numbers more or less defeat their own purpose because of the fact that, in the attempt to explain the new entirely in terms of the familiar, they so emphasize the illustrative objects that the attention is drawn away from the new process rather than being focused upon it. In other words, the thing being illustrated tends to become obscured by the illustration. It must not be forgotten that after all we are here *defining* certain operations with a totally new kind of numbers whose characteristics themselves depend upon arbitrary definition. Therefore, since we are thus extending our number system, obviously the new cannot be explained entirely in terms of the old. Rather, the main concern must be to show that the operations with the new (directed) numbers must be so defined that the operations with them are consistent with the old (arithmetical) operations.

As soon as the student has acquired an understanding of the nature of negative numbers as contrasted with positive numbers, he should be taught how to perform the fundamental operations with signed

numbers. The first of these operations to be undertaken is that of algebraic addition. Here the student is likely to experience some confusion in the beginning, due to the fact that in arithmetic the sum of two or more numbers is always greater than either of the addends. Reference to the number scale is useful in clearing up perplexities on this point. It should be clearly explained that henceforth "adding" will mean "combining" or "taking together." By reference to the

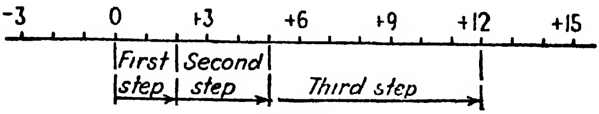

FIG. 17.

number scale it can be made clear why it is that, when a positive and a negative number are combined, the result will be less than the positive number alone.

The analysis of the addition (combination) of positive and negative numbers is entirely analogous to that of the addition of two or more positive numbers. For example, in finding the sum $2 + 3 + 7$, we start at the zero point on the number scale (Fig. 17) and count two

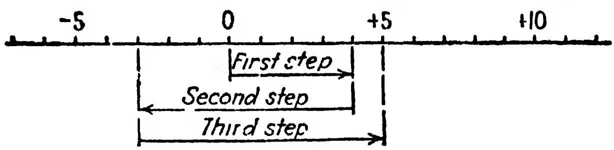

FIG. 18.

units to the right, since to the right is the direction in which we agree to count positive numbers. Then *from there* we count three more units to the right. Then *from there* we count seven more units to the right. The result of these operations leaves us at a point which is 12 units to the right of zero. Hence we say that the sum is $+12$. In a similar manner the sum $4 + (-7) + 8$ may be found. We start at the zero on the number scale and count 4 units to the right (Fig. 18). Then *from there* we count 7 units to the *left* (since this is the direction in which we agree to count negative numbers). Then *from there* we count 8 units again to the right. The result of these movements leaves us at a point 5 units to the right of the zero point on the number scale. Hence we say that the sum

$$4 + (-7) + 8 \text{ is } + 5.$$

Students will have little difficulty in seeing that the addition of negative units offsets or neutralizes a corresponding number of positive

units, or vice versa, so that the sum of any series of signed numbers is determined by seeing whether the series contains more negative units than positive or more positive units than negative, the absolute (or numerical) value of the sum of the series being given by the excess of the one over the other. From these considerations the student may formulate his own rule or method for determining the absolute value and the sign of the algebraic sum.

Simple and easy as this may appear, it requires much carefully supervised practice to fix these ideas and procedures firmly in the minds of the students and to give them assurance and facility in adding signed numbers. It must not be forgotten that this is a new and difficult extension of the students' mathematical experience and that they will require considerable time and experience to adjust themselves completely to it. It is particularly important that training in the addition of signed numbers must not be slighted or unduly hurried, for the reason that the concepts and procedures involved therein form the basis for understanding the subsequent processes of subtraction, multiplication, and division of signed numbers.

In the subtraction of one signed number from another, students may be expected to experience initial difficulty because they will need to revise their idea of the meaning of subtraction. In arithmetic, unless they have been taught to use the additive method, they will have come to regard subtraction as taking one number away from another, and, since arithmetic always deals with positive or absolute values, the result of subtraction as always less than the minuend. Now, however, it becomes necessary to analyze the process more carefully. The analysis should start with familiar arithmetical examples and then be extended to include both positive and negative numbers. For example, just what does it mean to subtract 5 from 8? Arithmetically it means to find what number is left when 5 units are taken away from an aggregate of 8 units. On the other hand, to speak about taking away "minus 5" units from an aggregate of units would have no meaning. Hence we must make a further and more general analysis of the meaning of subtraction. It is not difficult to point out to students that subtraction can be defined, in a more general way, as the process of finding what number must be added to 5 to give 8 and that, when we find this number, it will be precisely the number which is left when we take 5 away from 8. This definition has meaning when applied to either positive or negative numbers, since the student has already learned that they may be combined by addition. Similarly, the operation indicated by $12 - (-3)$ means that we must find

the number which added to (-3) will give 12. Obviously, this definition of subtraction will involve no new difficulties for students who have learned to use the additive method of subtraction in arithmetic.

This may be further illustrated by use of the number scale. Thus, since we are to find what number must be added to (-3) to give $(+12)$, we must start at (-3) and count to $(+12)$. This necessitates

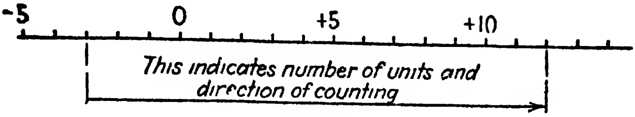

FIG. 19

moving over 15 units on the scale, since the direction of motion is to the right, the sign of the result is positive and the result is $(+15)$. In other words, $(-3) + (+15) = (+12)$, and this is merely another way of saying $12 - (-3) = (+15)$.

Again, take the example $5 - 8$. Here we must count from $(+8)$ to $(+5)$. That is, we pass over 3 units on the scale, but this time the direction of motion is to the left, hence the result is (-3). In other words, we have the result

$$\begin{array}{cccccc} \text{Subtrahend} & + & \text{result of subtraction} & = & \text{minuend} \\ (+8) & + & (-3) & = & (+5) \end{array}$$

which is another way of saying that $(+5) - (+8) = (-3)$.

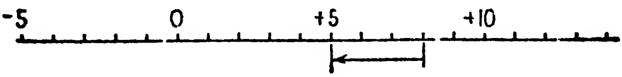

FIG. 20

Numerous similar examples exhibiting various combinations of positive and negative numbers should be given, some being explained by the teacher and others being worked out by the students, until the students thoroughly understand the process. As soon as the process is thoroughly understood, the examples should be reviewed, or others given. The students should then have their attention specifically directed to the fact that every subtraction, when analyzed in this way, gives a result which is the same as it would be if the sign of the subtrahend were changed and the problem treated as a problem in addition instead of subtraction. When this point has been made clear, it may be formulated into the rule. *To subtract one signed number from another change the sign of the subtrahend and then treat the problem as an addition problem rather than as a subtraction problem.* Henceforth the students should be expected to use this method of subtraction. Its

advantages are manifest because it reduces the process of subtraction to the already familiar process of addition. The rationalization of the method is not intended to be used after the method is understood but is solely for purpose of showing that this method is consistent with the meaning of subtraction. In spite of the apparent simplicity of the method the students should be given much closely supervised practice in its use.

In the multiplication of directed numbers the law of signs is very simply stated, but its rationalization can be rather complicated. Some writers have gone so far as to recommend the omission of any effort at rationalization. They would have the teacher merely state the rule and leave the understanding to come through use. Such procedure does not satisfy the curiosity that develops in the minds of many students as to why the rules are what they are.

One of the most effective methods of rationalization is to explain multiplication as repeated addition if the multiplier is a positive number. Thus $(+2)(+3)$, to be read $(+2)$ *multiplied by* $(+3)$, means $(+2) + (+2) + (+2) = +6$, and $(-2)(+3)$ means $(-2) + (-2) + (-2) = -6$. Attention should be called to the fact that $(+2)(+3)$ may also be written $(+3)(+2)$, read $(+3)$ *times* $(+2)$, and still mean to add $(+2)$ three times. Similarly, $(-2)(+3)$ may be written $(+3)(-2)$ to indicate the sum of $(-2) + (-2) + (-2)$. Each of these processes can be demonstrated very simply on the number scale.

Since the sign of subtraction is $-$, $(+9) - (+2)$ means that $+2$ is subtracted from $+9$ one time. The result is $+7$, which is the same result obtained from adding -2 to $+9$, or $(+9) + (-2)$. In a similar way $0 - (+2)$ is the same as $0 + (-2)$. Since $+ (-2)$ means to add -2 one time and $(+3)(-2)$ means to add -2 three times, so $-(+2)$ means to subtract $+2$ one time and $(-3)(+2)$ means to subtract $+2$ three times. Just as $+(-2) = -2$ and $-(+3) = -3$, so $(-3)(+2) = -6$ and $(+3)(-2) = -6$.

After it has been established that

$$(+2)(+3) = +6 \qquad \text{and} \qquad (-3)(+2) = (+3)(-2) = -6$$

attention should be called to the fact that in finding the product of two numbers, *the change of the sign of one of the factors changes the sign of the product*. Opportunity should be provided through practice to become familiar with this fundamental rule of multiplication. The question should then be raised as to what the effect would be if in either of the multiplication examples

$$(-3)(+2) = -6 \qquad \text{and} \qquad (+3)(-2) = -6$$

the $-$ sign were changed to a $+$ sign. This provides the opportunity for observing that the familiar product

$$(+2)(+3) = +6$$

is obtained if the rule is applied. After a few practice exercises of this type the class is ready for the question: "What will be the effect if, in either of the multiplication examples

$$(-3)(+2) = -6 \qquad \text{and} \qquad (+3)(-2) = -6$$

the $+$ sign is changed to a $-$ sign?" The application of the rule gives

$$(-3)(-2) = +6$$

Practice exercises should then be provided for becoming familiar with this last case of multiplication with signed numbers. A few review exercises then should lead to the summarizing of the four cases into the two rules:

1. *The product of two numbers which have like signs is positive.*
2. *The product of two numbers which have unlike signs is negative.*

Opportunity should then be given for practice in the application of the rules to the finding of the product of both arithmetical and literal numbers. Emphasis should be given to the fact that the numerical value of the product can be obtained by disregarding the signs, and the proper sign can be given to the product through application of the rules.

Since the quotient obtained by dividing one number by another is a number whose product with the divisor must give the dividend, it must follow that the law of signs for division is the same as that for multiplication. That is, if the signs of the divisor and the dividend are alike, the quotient will be positive, and, if they are different, the quotient will be negative. This explanation of the law of signs for division is usually quite satisfactory if illustrated by numerical examples, and ordinarily there is no material value in more elaborate attempts to rationalize it.

Exercises

1. Examine a recent set of textbooks in seventh- and eighth-grade mathematics, and list all the ideas or processes that you find in these books which are drawn from algebra. What things, if any, do you think should be omitted from this list, or added to it?

2. Make a list of simple formulas suitable for use in the seventh and eighth grades.

3. Enumerate the precise understandings or skills you would hope to have seventh- or eighth-grade students acquire from their study of formulas.

4. Contrast the purpose, nature, and content of the ninth-grade course in algebra with that of the algebraic work in the seventh and eighth grades.

5. If a student should ask you "What *is* a formula?" what answer would you give him? How would you distinguish between formulas and equations; between equations and identities?

6. Select from current newspapers or periodicals two illustrations of dependence in each of which a mathematical law is involved. In each case state the law in words and also as a formula.

7. How would you make clear to a class the distinction between constants and variables; between independent and dependent variables? Use numerous illustrations to give concreteness to your discussion.

8. For what particular reasons does the formula seem to be the most suitable avenue for introducing students to the study of algebra?

9. Make a list of the specific outcomes at which you would aim in teaching literal numbers and formulas to a class in ninth-grade algebra. In what respects does this list differ from the one you made under exercise 3?

10. Make a diagnostic test to use after teaching the unit on literal numbers and formulas outlined in exercise 9. Prepare also a scoring key and a tabulation sheet to use with this test.

11. Give a good review of Everett's book on "The Fundamental Skills of Algebra" (see Bibliography), pointing out the distinction which he makes between "manipulative skills" and "associative skills," and giving illustrations of each.

12. Excellent discussions of teaching the solution of simple linear equations are given in Ligda's book and in the book by Hassler and Smith (see Bibliography). Review the discussions by these authors, and compare them with the suggestions made in this book.

13. Outline and justify those topics you would emphasize in teaching simple linear equations to ninth-grade students.

14. Select a variety of simple linear equations in one unknown, and go through the solutions of these individually and in detail, showing how your plan provides an effective and simple but general method of attacking any of these variations of the simple linear equation.

15. What are the detailed advantages which students can derive from checking their own work in algebra?

16. Explain how the checking of solutions of equations gives good training in precisely the same kinds of mathematical activities as those used in the evaluation of formulas. Enumerate these in detail.

17. Simple linear equations are often first presented and taught under various "type forms." Similarly, problems are often classified and studied by types. What arguments could be advanced in support of this practice? What disadvantages could it have? What do you think of it?

18. For each of the following make an analysis of the understandings and skills which students would need to perform the tasks indicated, and by reference to your lists point out the specific difficulties which you would expect normal students to encounter: (*a*) translating rules into formulas; (*b*) translating formulas into rules; (*c*) evaluating formulas; (*d*) deriving new formulas from given formulas; (*e*) making graphs to represent formulas; (*f*) interpreting graphs of formulas.

19. The same rules of operation are employed in solving formulas and literal equations as are used in solving ordinary simple equations in one unknown; yet

students experience more difficulty in solving formulas than in solving ordinary simple equations. Why is this, and what, if anything, can be done about it? Be specific.

20. Exactly what is meant by *formalism* in algebra? Exactly what is meant by the *formalization* of algebraic processes?

21. Criticize or support the assertion that the very essence of algebra lies in the generalization of its concepts and the formalization of its processes.

22. Is the "transposition of terms" an algebraic process? Should it be taught as a process and under this name in ninth-grade algebra? Why or why not?

23. Discuss the proposition that in teaching the operations with directed numbers there may be danger in overrationalization as well as in overmechanization.

24. How would you explain to a class in ninth-grade algebra the principle that $(-a)(-b) = ab$?

25. Make a list of specific difficulties which students encounter in learning to work with directed numbers. Make a thorough analysis, and be very specific.

26. Take two of the difficulties which you listed in the preceding exercise, and describe in detail how you would go about helping students avoid or overcome them.

27. Which is the more important to the average student from the standpoint of later application, the ability to construct graphs or the ability to read and interpret graphs readily? Does the one imply the other? Discuss the pedagogical implications of your answers to these questions.

28. Try to find in current newspapers, periodicals, or books both good and poor examples of each of the following kinds of graph: (*a*) bar graph; (*b*) broken-line graph; (*c*) circle graph. Criticize each of your examples favorably or unfavorably.

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CHAPTER XIII

FURTHER TOPICS IN NINTH-GRADE ALGEBRA

Teaching the Solution of Simultaneous Linear Equations. The five commonly used methods of solving simultaneous linear equations are (1) the graphical method, (2) the method of elimination by substitution, (3) the method of elimination by addition and subtraction, (4) the method of elimination by comparison, and (5) the method of determinants.

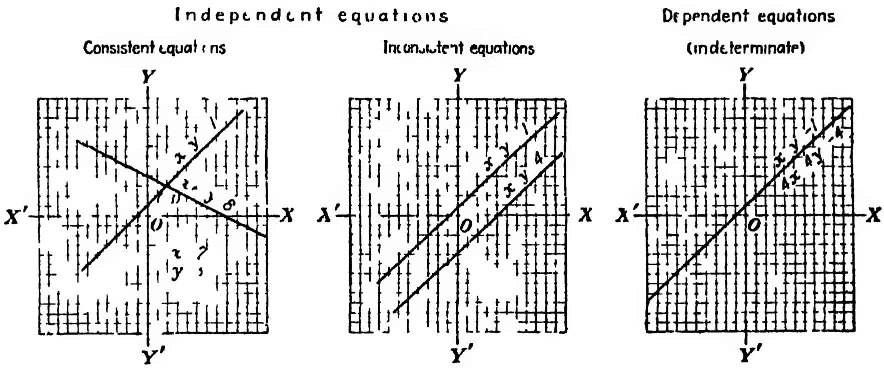

Fig. 21.

The graphical method can be used, of course, only with students who have previously studied the meaning and construction of graphs of linear functions. Its principal advantages lie in the fact that it is interesting and that it illustrates in a very convincing manner the reason why a solution of such a system must consist of a pair (or set) of numbers rather than of a single number. It affords a very effective method for demonstrating the full significance of the relationships that exist when the equations are consistent, inconsistent, indeterminate, or dependent (see Fig. 21). This method also proves valuable in developing a clear understanding of just what is meant by *simultaneous linear* equations. Auxiliary advantages are that it gives an excellent review of graphs of linear functions and of the associated concepts and procedures.

Its main disadvantages are two. In the first place it is possible,

in general, to get only approximate solutions instead of exact ones. This often introduces apparent discrepancies in checking the solutions and gives the student a consequent feeling of dissatisfaction. Secondly, it is a comparatively slow, tedious, and inefficient method for solving simple linear equations. For this reason, after students have learned the more exact and efficient algebraic methods, they are likely to prefer them to the graphical method.

Aside from any difficulties which the students may experience in the construction of the graphs themselves, the only point of potential difficulty involved in the graphical method is the interpretation of the solution. To assist students in this, the teacher should remind them that any point on any graph has two coordinates (numbers) associated with it, and that these numbers satisfy the equation which was used in making the graph. Therefore, if a point *lies on two graphs at the same time* (say graph A and graph B), its coordinates must satisfy the equation used in making graph A and at the same time they must satisfy the equation used in making graph B. Consequently the coordinates of this point must form a solution of the system. This basic concept is the real crux of this method of solving simultaneous equations, and, unless it is strongly emphasized by the teacher, the students may miss the main point of the whole procedure.

If the students have developed a thorough understanding of the meaning of substitution and of how to solve and evaluate formulas and literal equations, there is nothing new for them to learn in solving simultaneous equations by the method of substitution. Consider, for example, the system

$$\begin{aligned} 3x - \quad y &= -13 \\ 2x + 3y &= \quad 17 \end{aligned}$$

The first step is to solve one of the equations for one of the unknowns in terms of the other. This is merely the solution of a literal equation or formula. The student should select the one which can be solved most easily. In this case he would probably solve the first equation for y and get the result $y = (3x + 13)$.

He will now substitute this result for y in the second equation and get a resulting equation in which no term in y appears:[1]

$$\begin{aligned} 2x + 3(\quad y \quad) &= 17 \\ 2x + 3(3x + 13) &= 17 \end{aligned}$$

[1] Note the use of parentheses.

Next he will perform the indicated operations and collect terms:

$$\begin{aligned} 2x + 9x + 39 &= 17 \\ 11x \qquad + 39 &= 17 \end{aligned}$$

Then he will solve this equation for x:

$$\begin{aligned} 11x \qquad + 39 &= 17 \\ 11x &= 17 - 39 = -22 \\ x &= \frac{-22}{11} = -2 \end{aligned}$$

Finally he must substitute this value of x back in the first equation (the one which he originally solved for y) to get the numerical value of y:

$$\begin{aligned} y &= 3(\ x\ ) + 13 \\ y &= 3(-2) + 13 = -6 + 13 = +7 \end{aligned}$$

The solution of the system has now been completed, but it should be checked in both of the original equations. The first equation is as follows:

$$\begin{aligned} 3(\ x\ ) - (y) &\stackrel{?}{=} -13 \\ 3(-2) - (7) &\stackrel{?}{=} -13 \\ -6 \quad - 7 &= -13 \end{aligned}$$

This checks. The second equation will be checked in the same manner.

The detailed steps in the foregoing illustration have been given for the purpose of showing how the principles and procedures discussed in the previous chapter are now applied by the student to this new situation involving a system of simultaneous equations. In particular it is to be noted that the student does not need to learn or to use in this process anything that is new to him He has merely to learn to use familiar mathematical tools in a new setting In the process of doing this he receives a most valuable review, a fuller comprehension of the nature of these mathematical tools, and at the same time gains added proficiency and facility in their use [1]

The method of elimination by addition and subtraction is very effective if the method of substitution leads to the substitution of fractions, to a complicated combination of literal numbers, or to a maze of parentheses or signs of aggregation. It is a very simple process and one that is rather easily extended later in the work of the senior high school and junior college to systems of equations in any number of unknowns.

[1] In connection with the detailed steps in the foregoing illustration, the reader may well refer to the sections of the previous chapter dealing with literal numbers and formulas and the solution of equations.

The method of elimination by comparison and the determinant method of solving simultaneous equations are very effective but are generally reserved for more advanced work.

Operations with Fractions. The ordinary operations with fractions which find a legitimate place in ninth-grade algebra include reduction and "stepping up," multiplication and division, addition and subtraction of fractions. The student's experience with arithmetical fractions will serve as a point of departure for beginning the work with algebraic fractions. It would be a mistake, however, to assume that familiarity with arithmetical fractions will eliminate all difficulty in working with algebraic fractions. Algebra lacks the familiar, concrete, intuitive basis which characterizes a good deal of arithmetic, and the student must learn eventually to work almost entirely upon the basis of established rules or patterns of procedure rather than by intuitive methods. These patterns, of course, are merely generalizations of the methods used in arithmetic and so may be developed and explained largely by means of analogy with arithmetical situations. The chief difference to be emphasized is that in algebra the student will find it necessary to confine his attention more and more specifically to the processes by which he works and to pay relatively less attention to the particular numerical values of the quantities involved.

Since the use of fractions in elementary algebra is confined largely to the solution of simple formulas and certain types of verbal problems and to the simplification of algebraic expressions, it is neither necessary nor desirable to include highly complicated fractional expressions in the work of the ninth grade.

Many textbooks, both in arithmetic and algebra, introduce the addition and subtraction of fractions before taking up the multiplication and division of fractions. There are those who contend that this plan is psychologically unsound. They argue that multiplication and division are the less difficult of the operations with fractions and, hence, should be studied first. In accordance with this point of view the plan of organization followed in some of the more recent texts is to consider first the reduction and "stepping up" of fractions, then the multiplication and division of fractions, and finally the addition and subtraction of fractions.

The student should acquire through his study of fractions the following abilities and understandings:

1. Understanding of the four aspects of the meaning of a fraction
2. Understanding of what it means to reduce a fraction to lower terms or to raise a fraction to higher terms

3. Understanding of the nature of the terms of a fraction, and how the increase or decrease of the numerator or denominator affects the value of the fraction
4. Ability and skill in multiplying fractions together or in dividing one fraction by another
5. Ability and skill in adding or subtracting fractions

In arithmetic various meanings are attached to the word "fraction." Thus

A fraction is one or more of the equal parts of a unit. . . .
A fraction is one of the equal parts of a single quantity consisting of one or more units. . . .
A fraction is the quotient which results from dividing one number as the dividend by another, the divisor, the value of the fraction *in toto* being the actual value of the quotient. . . .
A fraction is the value of the ratio which one number (the numerator) bears to another number (the denominator). . . .[1]

The idea which seems to persist most vividly in the minds of most students is that a fraction is a part or a number of parts of some quantity. This is doubtless because many arithmetical fractions, and even operations with such fractions, can be most easily and concretely illustrated by reference to tangible objects, visible geometric or graphic diagrams, and denominate numbers. Thus the student acquires a sort of intuitive feeling for a fraction as a part of something. In algebra, however, the precise numerical relationships are not present in the sense that they are in arithmetic; hence it becomes necessary to construct a less intuitive but more definitive meaning. The student must now learn to think consistently of a fraction as merely an indicated quotient or an indicated division of one quantity (the numerator) by another quantity (the denominator). Whether the indicated division can actually be carried out exactly or not is immaterial

If students are to learn to attach this meaning to a fraction, it will be necessary for the teacher to give specific attention to it and to focus the attention of the students upon it repeatedly. It may be illustrated and made to appear reasonable by reference to concrete, numerical, geometric, or physical quantities, but it must not stop with such illustrations. Illustrations must also be given in which the precise numerical relations which provide the intuitive basis for the earlier meaning of a fraction are replaced in part or in their entirety by purely symbolic quantities, and students must be specifically trained to regard such

[1] Harrier E. Glazier, "Arithmetic for Teachers" (New York: McGraw-Hill Book Company, Inc., 1932), pp. 110–112.

expressions as a/b or $(x - 3)/2y$ as being fractions just as truly as $3/4$ or $8/15$. The meaning itself must be made to have the status of a definition, so that it may persist even in cases where no intuitive basis exists.

The *reduction of fractions*, as well as the inverse operation of raising fractions to higher terms, involves the general problem of changing the form of a fraction without changing its value. Such changes can always be effected by the operation of a single simple principle, *viz.*, that *the value of a fraction remains unchanged if its numerator and denominator are multiplied or divided by the same quantity.*[1] Children generally are able to apply this principle quite successfully when working with arithmetical fractions, but it is probably done intuitively and without much conscious recognition of the principle itself. Evidence of this is found in the fact that, when they come to work with algebraic fractions, they often fail to apply the principle and consequently get such erroneous results as $\frac{x+4}{4} = x$ or $\frac{2n+a}{-3n+a} = \frac{2}{-3}$. Results of this sort are almost certainly due to the failure of the students to realize that this procedure does not constitute division of numerator and denominator by a common factor. The undefined use of the word "cancellation" probably contributes to this looseness of thinking in many cases.

Students must be kept aware of the fact that the addition or subtraction of like quantities (except zero) to or from the numerator and denominator of a fraction will certainly change not only the form but the value of the fraction, whereas in the reduction of fractions the value of the fraction must be kept intact. They must also be kept aware that in dividing numerator and denominator by a common factor, the factor must be a divisor of the *whole* numerator and of the *whole* denominator. Continued consciousness of this elementary principle will prevent the occurrence of such errors as this

$$\frac{3a+5}{a+2} = \frac{3+5}{2} = \frac{8}{2} = 4.$$

Most of the mistakes which occur in the reduction of fractions could be avoided by the consistent practice of expressing (rewriting if necessary) the numerator and denominator of the fraction in factored form and of enclosing each separate factor, no matter how simple, in its own parentheses. If this is done, the division or "cancellation" of com-

[1] Division by zero is, of course, excepted, and here multiplication by zero must also be excepted since it would render the value of the fraction indeterminate.

mon factors will be effected with real understanding of what is being done. Similarly, in raising fractions to higher terms, it is good practice to have the students immediately enclose the numerator and denominator of the original fraction in parentheses at the outset and then write the common multiplier also in parentheses as a multiplier of the numerator and as a multiplier of the denominator. The consistent use of parentheses in this way serves to keep all factors intact and to prevent students from treating a part of a factor as a whole factor.

The *multiplication of fractions* generally causes little difficulty. The principle governing this process is exceedingly simple: the product of two or more fractions is a fraction whose numerator is the product of the numerators of the original fractions and whose denominator is the product of their denominators. It should be illustrated freely by examples drawn from arithmetic, and, by analogy, the application to algebraic expressions can be made without difficulty.

The fractional product can often be reduced to lower terms. Hence it is desirable to have it rewritten as a single fraction, the numerators and denominators of the component fractions being written in factored form with every factor enclosed in its own parentheses. It should be stressed that the numerator of every component fraction must be regarded as a factor of the numerator of the product, and similarly for the denominators. The students will be more likely to keep this consciously in mind if they make a habitual practice of enclosing the individual numerators and denominators of the component fractions in parentheses at the outset, before anything else is done. Thus

$$\frac{4 - x^2}{2 + x} \cdot \frac{x}{2 - x}$$

would be rewritten as $\dfrac{(4 - x^2)}{(2 + x)} \cdot \dfrac{(x)}{(2 - x)} = \dfrac{(2 - x)(2 + x)(x)}{(2 + x)(2 - x)} = x.$

In the *division of one fraction by another* it is necessary only to see that the students understand that division is always exactly equivalent to multiplication by the reciprocal of the divisor. They will probably be more or less familiar with this principle from their arithmetic, especially as regards division by a fraction. However, the principle should be clearly explained and numerous illustrations given. The student should understand that dividing by a fraction means finding the number which, when multiplied by the divisor, gives the dividend. This is always the test of division. For example, if 7 is divided by $\frac{3}{5}$ the result must be such that the product of itself and $\frac{3}{5}$ will give 7 [or 7(1)]. We must therefore find a number q such that $q(\frac{3}{5}) = 7$. Evidently

q must equal $7(5/3)$, since $7(5/3)(3/5)$ equals $7(1)$ or 7. Thus 7 divided by $3/5$ equals 7 times $(5/3)$. Similarly, in the example $a \div (x/y)$, we have $a \div (x/y) = q$. But $q \cdot (x/y) = a$, whence $q = a \cdot (y/x)$. Thus

$$a \div x/y = a \cdot (y/x)$$

As a rule the main difficulties to be anticipated are those incident to the multiplication and the simplification of fractions, and these have been discussed in the foregoing paragraphs.

Sometimes students have difficulty in multiplying or dividing fractions by whole numbers. Some writers prefer to treat this as a special case to be covered by the rule: *in multiplying or dividing a fraction by a whole number only the numerator of the fraction is to be multiplied or divided by the whole number.* Others feel that it is better to have the students regard the whole number as the numerator of a fraction whose denominator is 1, thus bringing the problem under the general procedures for multiplying and dividing fractions. Either of these methods will probably give satisfactory results. The second, however, appears to have certain advantages over the first, in that it gives more unity and generality to the whole matter of multiplying and dividing fractions and obviates the necessity for the special rule. It also has the advantage of applying to the division of a whole number by a fraction, which is not precisely covered by the special rule.

In introducing students to the *addition of fractions* care must be taken to prevent the occurrence of such errors as this:

$$\frac{5x}{a} + \frac{2}{n} = \frac{5x + 2}{a + n}$$

Mistakes of this kind reveal a lack of understanding of the real nature of a fraction and of the real meaning of the numerator (numberer) and the denominator (namer). The students should be kept conscious of the principle that only things of the same kind can be combined by addition or subtraction and that fractions are considered to be of the same kind only if they have like denominators. If students can really get the idea that the denominator of a fraction merely indicates what kind of things are being considered and that the numerator merely indicates how many of these are taken, they will have gone far toward heading off mistakes of the kind described above. Appropriate illustrations of arithmetical fractions are helpful in emphasizing this point because, if sufficiently simple illustrations are taken, the students can immediately check the correctness of their results. For example, they

know that $\frac{3}{4} + \frac{1}{4} = 1$, and they can at once see the discrepancy of saying that $\frac{3}{4} + \frac{1}{4} = \frac{3+1}{4+4} = \frac{4}{8} = \frac{1}{2}$, because $\frac{1}{2}$ is obviously not equal to 1, and they know that the correct answer must be 1. They should be trained to the habit of using some such simple numerical illustration to check up any doubtful operation to see whether or not it is legitimate and correct, and they should learn also to go back to the original meanings of the numerator and denominator to clarify their thinking. To this end it is often helpful to rewrite fractional expressions with the denominators written out in words. Thus $\frac{2}{7} + \frac{3}{7}$ can be written as 2 *sevenths* + 3 *sevenths*, in which case the expression becomes similar to the expression for the sum of any like denominate quantities such as 2 *dollars* and 3 *dollars*. In other words, this process helps to make clear the idea that fractions are in a sense the same as denominate numbers, the denominator merely telling what kind of thing is being considered and the numerator telling how many are being considered.

The tendency to add numerators and to add denominators is probably a carry-over from the operation of multiplying fractions. It is important that the students get the distinction between these processes clearly in mind so that they will not confuse them.

The first step in teaching students to add fractions is to have them recall the basic principle of addition: Addition can be used only for combining groups of like things. Thus, fractions cannot be added until they have been changed to fractions with a common denominator, for then, and then only, are they groups of like things. The denominator of the sum is the common denominator of the addends, and the numerator is the algebraic sum of their numerators. It is not sufficient, however, merely to tell students this. The teacher should give numerous carefully selected illustrations, working them out at the blackboard and discussing them with the class while he works. The process is not difficult either to understand or to perform, but it needs to be explained very carefully and deliberately and to be fixed very carefully in the minds of the students.

Such careful and adequate explanation can give the students an understanding of the addition of fractions, but the fixation of the ideas and the procedure requires that the students shall have a substantial amount of practice in doing the thing themselves. The practice exercises should be selected with great care. They should start with very easy exercises and proceed only by gradual stages to the more difficult

ones. The increasing complexity will be essentially in the numerators, since in this first stage only fractions having like denominators will be used. There should be no great difficulty if care is taken, because the only real algebraic work will be the algebraic addition of the numerators. This will be essentially nothing but a review of the algebraic addition of polynomials, with which the students should be expected to have considerable familiarity. The exercises should include fractions whose numerators are of varied types including simple integers, literal monomials, binomials, and trinomials, some of the binomial and trinomial numerators being given in factored form requiring expansion in order to effect the simplification of the resultant sum.

Too much emphasis cannot be laid upon the principle that the transition to the more complicated forms should be made *gradually*. Teachers often assume that little attention need be given to the addition of fractions with like denominators. No greater mistake could be made. This assumption is entirely unwarranted. The fact is that, when the students *really understand* the addition of fractions with like denominators, they are likely to have little difficulty in mastering the addition of fractions with unlike denominators.

When fractions with unlike denominators are to be added algebraically, the students should be firmly impressed with the guiding principle: *if the denominators are not alike, make them alike.** That is, the students should become clearly aware that the first thing to do is to change the given fractions into new fractions whose values are identical with those of the original fractions but whose forms are changed in such a way that they will all have the same denominator. After this has been done, the problem of the algebraic addition of these fractions becomes the already familiar one which has been described above.

While the statement of this principle sounds simple and clear, there are often intermediate details which will tend to obscure the essential simplicity of the matter. Consider, for example, the following case: $\frac{3a}{5n+2}+\frac{5a}{2n-3}$. Here it is necessary to determine first what the common denominator will be. Since neither denominator is a factor of the other, the common denominator will have to be taken as the product of all (in this case both) of the separate denominators, *viz.*, $(5n+2)(2n-3)$. This should always be expressed in factored form. Now, in order to make the first fraction have this common denominator, it is necessary to multiply its denominator (and consequently its numerator too) by the factor $(2n-3)$. Similarly the denominator

and numerator of the second fraction must be multiplied by the factor $(5n + 2)$. The two fractions now have the form

$$\frac{3a(2n - 3)}{(5n + 2)(2n - 3)} + \frac{5a(5n + 2)}{(2n - 3)(5n + 2)}$$

in which the denominators are alike. The student may proceed to write the sum by writing the product $(5n + 2)(2n - 3)$ as the denominator of the sum, $\frac{}{(5n + 2)(2n - 3)}$, and then writing in the numerator the indicated sum of the numerators of the transformed fractions: $\frac{3a(2n - 3) + 5a(5n + 2)}{}$. Thus the sum will be written as

$$\frac{3a(2n - 3) + 5a(5n + 2)}{(5n + 2)(2n - 3)}$$

The student should feel that, when he has carried the work this far, he has already completed the essential part of the job of adding the fractions, although it may often be desirable to simplify the expression which he has obtained. Generally this simplification will consist merely of expanding the indicated products and collecting like terms. Occasionally it may require a refactoring of the numerator and denominator and the "cancellation" of factors common to both. However, most problems in the addition of fractions in first-year algebra should be comparatively simple. Certainly any types which exceed in difficulty the illustrative example discussed above should be reserved for subsequent courses.

Teaching the Solution of Equations Containing Fractions. Students often have difficulty in solving equations containing fractions even though they may readily solve equations without fractions. The difficulty generally can be traced to the complicated appearance which the presence of fractions give. to the equation. Lacking experience with such equations, the student tends to become confused at the outset because he does not know how to start the analysis of his problem.

The student needs to be taught two basic things to help him out of his difficulty. (1) He must come to understand that equations which contain fractions may be changed into equivalent equations which do not contain fractions and that, when so changed, equations in one unknown may be solved readily because they will then be in the same form as the equations to which he is accustomed. (2) He must learn how to change an equation containing fractions into an equivalent equation which does not contain fractions. After he has done this, he

has merely to deal with ordinary linear equations concerning which suggestions have been given in the preceding chapter.

A few illustrations should serve both to convince him that it is in general possible to change an equation containing fractions into an equivalent one which does not contain fractions and to make clear to him the method by which this is accomplished. The method of explanation should be substantially as follows:

Let us consider the equation $\frac{2}{3} + \frac{3}{x} = \frac{5}{6}$, in which we are required to solve for x. We know how to solve equations without fractions; hence, if we could change this equation into one that would contain no fractions, we could solve it.

We could do this, if we could get rid of the denominators. But the only way we can get rid of them is to have a factor in the numerator of each term which is exactly equal to the denominator of that term, so that we may divide both numerator and denominator of each individual term by the whole denominator of that term.

We can get new factors in the numerators of all the terms if we multiply every term in the whole equation by the same quantity, because multiplying a fraction means multiplying its numerator. We have a right to multiply all the terms in the equation by any common multiplier we wish, because while this changes the value of each individual term it does not destroy the equation.

We wish, therefore, to find the smallest multiplier which can be exactly divided by the denominator of each fraction in the equation This multiplier will be, as you know, the least common denominator of all these fractions. In the case of the equation which we are considering in this problem, the L.C.D. will be $6x$, since this is the smallest quantity of which 3, x, and 6 are all exact factors.

Let us therefore multiply each term in the equation by $6x$. This gives $\frac{6x(2)}{3} + \frac{6x(3)}{x} = \frac{6x(5)}{6}$.

Now, if we reduce each term to its simplest form by dividing its denominator and numerator by whatever factor is common to both, we have a resulting equation $4x + 18 = 5x$, which has no fractions in it. We can easily solve this equation, since we have solved many others like it.

Students often make mistakes in solving equations containing fractions because they are careless in writing their work down. With special frequency they produce crowded and often illegible work when inserting the L.C.D. as a common multiplier of the numerators of the

various terms in fractional equations. The following illustration is not exaggerated:

$$\frac{n}{2n-8}+\frac{16}{n^2-16}=\frac{1}{2}; \qquad \frac{n}{2(n-4)}+\frac{16}{(n-4)(n+4)}=\frac{1}{2}$$

$2(n-4)(n+4)$ $\qquad$ $2(n-4)(n+4)$ $\qquad$ $2(n-4)(n+4)$

$$\frac{n}{2(n-4)}+\frac{16}{(n-4)(n+4)}=\frac{1}{2} \text{ L.C.D } = 2(n-4)(n+4)$$

Such carelessly written work can hardly fail to be productive of mistakes. It can be avoided if the students are taught to do a little planning with regard to the details of form. They should learn to recognize that, where factors are to be inserted, space will be required for writing these factors and that they should provide such space in order that their work will be neat and legible. Thus the foregoing problem might be advantageously written as follows:

$$\frac{n}{2(n-4)} + \frac{16}{(n-4)(n+4)} = \frac{1}{2}$$

$$\frac{n(\cancel{2})(\cancel{n-4})(n+4)}{\cancel{2}(\cancel{n-4})} + \frac{16(2)(\cancel{n-4})(\cancel{n+4})}{(\cancel{n-4})(\cancel{n+4})} = \frac{1(\cancel{2})(n-4)(n+4)}{\cancel{2}}$$

$$n(n+4)+32=(n-4)(n+4), \text{ etc.}$$

Where the need for space in writing is anticipated and provided for in this way, the written work is invariably improved in neatness and legibility, and the likelihood of mistakes is greatly diminished. Although this is a matter of form rather than of mathematics, it is by no means a trivial matter.

Sometimes students confuse the procedures involved in solving equations containing fractions with those involved in the addition of fractions. It is important to point out clearly that the two problems are fundamentally different. In the one case the aim is to find the algebraic sum of certain given fractions. This obviously makes it necessary to preserve the *original value* of each individual fraction although its form may be altered. On the other hand, in solving an equation, both the form and the value of the individual terms may be altered if necessary, provided that the *equality* of the two members of the equation be preserved. For this reason, in adding fractions, we may not multiply either the numerator or denominator of any fraction unless we multiply *both* by the same factor, whereas, in clearing an

equation of fractions, we have a perfect right to multiply the numerators of *all* the terms by *the same multiplier* without multiplying the denominators at all. It is important that illustrations of this point be presented and discussed with the students and that they shall come to make a clear distinction between the meanings and the implications of these two fundamentally different problems.

The Solving of Verbal Problems. The solving of verbal problems is one of the most troublesome parts of algebra for most students. In general, such problems involve relationships which may be cast in the form of one or more equations, and those problems which are found in algebra textbooks ordinarily involve relationships that can be represented by relatively simple equations which can be solved without any difficulty once they are set up. The trouble lies in setting up the equations, *i.e.*, in translating the verbal statements into algebraic language. Therefore the principal effort of both teacher and students in connection with the study of verbal problems should be directed primarily toward developing the ability to translate the problems into equations.

The difficulty which children encounter in making such translation is quite understandable. In the first place, most children are not very careful analytical readers. The recent emphasis in the schools on rapid cursory reading is doubtless appropriate and valuable for many purposes, but it does not lend itself well to the careful analysis of problems. Analysis is characteristically a slow and tedious process, and the ability to read analytically requires patience as well as concentrated and sustained attention. These characteristics, generally, can be developed to a satisfactory degree only by special training in giving conscious attention to them. The teacher of algebra must assume responsibility for giving this special training in careful analytical reading if he expects to have his students become proficient in solving verbal problems.

One of the material difficulties encountered by children in interpreting the verbal statements of problems lies in the fact that the relationships are not always stated explicitly but are often *implied.* For example, in the familiar distance-rate-time problems relationships between units of measure are implied, such as the number of feet in a mile or the number of seconds in a minute. In the many problems that imply the use of money, the relationships between the various units of monetary value are always assumed as known. Such words as "complementary," "supplementary," "right triangle," and numerous others are used to imply pertinent facts or relationships which are not stated explicitly. Hidden implications of this sort may appear so

obvious to the teacher that he will perhaps not even be consciously aware of the lack of explicit statement. Often, however, they constitute a real source of confusion to students. The students need to be trained specifically to be on the lookout for these hidden implications, to learn to detect them, and to take them into account in analyzing and setting up the problems.

Closely associated with the difficulties involved in this implicit manner of indicating facts or relationships is the difficulty which sometimes arises from the use of words or expressions whose meanings are not entirely clear to the students. For example, the expression "x less 5" means "$x - 5$," while the expression "x less than 5" means "either $5 - x$ or $x < 5$," which is a very different thing, although students frequently fail to detect the difference because they do not recognize that the meanings to be attached to the word "less" are not the same in the two expressions.

In the second place, the careful analysis of problems requires much patience, concentrated attention, and the willingness to take the time to write down and organize all relevant data with painstaking care. These are not generally to be regarded as normal characteristics of normal healthy young children. Children tend to be impatient with problems which they cannot organize intuitively in a moment. They want to get to the answer quickly and are often content to dismiss, with the remark "too hard," any problem involving relationships that cannot be seen and organized at a glance. Development of the ability to give concentrated and sustained attention is not only desirable as a general trait but it is absolutely necessary in the successful study of mathematics, and it is especially necessary in setting up verbal problems. The students need to be made and kept specifically conscious of this fact and to be trained in the habits implied.

Finally, and perhaps most important of all, students have difficulty with verbal problems because there is no single general pattern according to which all verbal problems can be set up. Certain "general methods" have been proposed and are doubtless helpful in systematizing the analysis, but there is, and can be, no formula which will obviate the necessity for alertness, care, ingenuity, and resourcefulness on the part of the student. The solution of equations, the addition of fractions, and many other operations with symbolic algebraic expressions can be reduced to mechanical laws which operate invariably. It is not so with the setting up of verbal problems. Every problem presents its own peculiar elements, relationships, and requirements which must be studied, interpreted, and organized strictly and solely

in the light of the conditions and data stated or implied in the problem itself.

It is true that many of the problems customarily found in the textbooks tend to fall into certain general groupings or "types." It is also true that several of these "types" have characteristic formulas which express the relationships involved. To this extent verbal problems may be classified and their solutions somewhat standardized, and for this reason many authors both advocate and practice the procedure of presenting verbal problems according to type. On the other hand, the component elements and the mathematical relationships involved in one type of problem may be entirely unlike those involved in other types. Thus, while a student may be able to set up and solve type problems when he knows to what types they belong, he may be completely at a loss when he attempts to classify problems as to type. For this reason the advisability of teaching verbal problems by type seems, at best, questionable. It does tend to produce specific classroom results more quickly than any other method, but they are results which are more in the nature of specific skills rather than general powers. Some authors suggest compromising the situation by supplementing the type lists of problems by unclassified lists of miscellaneous problems, holding that this secures the advantages of teaching problems by type and at the same time avoids the disadvantages. It would seem that this might be an effective compromise provided that appropriate emphasis were given to the study of the unclassified problems.

In general, verbal problems lead to equations. Therefore, somewhere in the problem, it should be possible to find at least one quantitative element for which two different mathematical expressions can be obtained. This element may be a particular distance, a particular volume, or any one of a variety of quantitative elements. The equality may be expressed specifically, or it may be merely implied. The search for such an element and for the two different ways of expressing it constitutes the analysis of the problem.

How much alcohol must be added to a pint of 10 per cent solution of iodine to make an 8 per cent solution?

Here the quantitative elements in the problem are the amounts of alcohol, the amounts of iodine, and the total amounts of the solutions. If each of these is considered under both the initial and final conditions, it is seen that the element which remains quantitatively the same in both cases is the amount of iodine. Thus we get the basic equation:

Original amount of iodine = final amount of iodine

By considering now the composition of each solution and designating by a letter (n) the number of pints of alcohol to be added to the original solution, we may readily get two different expressions for the amount of iodine. A diagram will be helpful in this case.

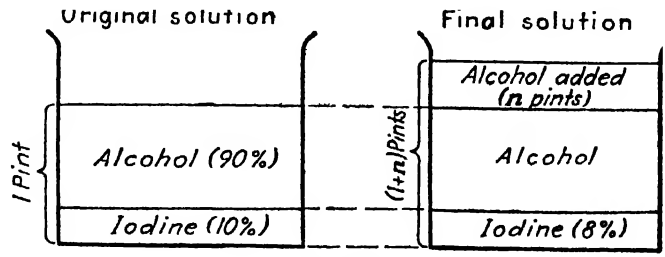

FIG. 22.

It is seen from the diagram that the original amount of iodine is $^{10}/_{100}$ of 1 pint and that the final amount of iodine is $^{8}/_{100}$ of $(1 + n)$ pints. By substituting these expressions in the basic equation given verbally above, we get the equation

$$\frac{10}{100}(1) = \frac{8}{100}(1 + n) \qquad \text{or} \qquad 10 = 8(1 + n)$$

from which the value of n can be found directly.

Two trains, 350 miles apart, travel toward each other until they meet. Train A travels at an average speed of 55 miles per hour and train B travels at an average speed of 48 miles per hour. How long will it be after they start before they meet?

Here the elements involved in the two situations are distances, rates, and times. One element, which is the same in the case of both trains, is obviously the time. Thus we may set up our basic equation in words:

$$\text{Time for train } A = \text{time for train } B$$

It remains now merely to get two mathematical expressions representing the time in terms of respective distances and rates of speed. To this end the detailed data should now be tabulated in some such manner as the following:

	Train A	Train B
Distance traveled, miles....................	x	$350 - x$
Rate, miles per hour........	55	48
Time (distance/rate)......................	$\frac{x}{55}$	$\frac{350 - x}{48}$

The substitution of these two expressions for the time in the basic equation given verbally above gives the desired equation in algebraic form:

$$\frac{x}{55} = \frac{350 - x}{48}$$

This may be solved for x, the distance traveled by train A, and, since the rate is known, the time can be easily computed.

Another, and perhaps easier, approach to this particular problem may be made by reflecting that the total distance is constant and that two expressions for this distance must be equivalent. One such expression is given explicitly in the problem, *viz.*, $d = 350$ miles. Another may be inferred from the fact that the total distance is the sum of the distances traveled by the two trains. Thus we have a basic equation:

Total distance = total distance

(Miles traveled by train A) + (Miles traveled by train B) = (350 miles)

Again tabulating the data, recalling that the trains travel the same number hours t and making use of the relation $d = r \cdot t$, we have

	Train A	Train B
Rate, miles per hour	55	48
Time, hours	t	t
Distance traveled, miles	$55t$	$48t$

Substituting now in the basic equation (above), we get the simple equation

$$55t + 48t = 350$$

from which the required length of time can be found directly.

This latter illustrative example makes it clear that there is considerable latitude in the analysis of many verbal problems and that success depends largely upon the care and ingenuity of the student. The setup of the problem may involve implied relationships which the student must seek in his background of experiences. The student who in his earlier work in arithmetic and informal geometry has accumulated a rich store of ready information about such relationships will have a great advantage in the analysis of problems. It is often possible to make graphic or diagrammatic sketches which are helpful in

making the relationships among the elements seem concrete and tangible.

The following suggestions, while they neither constitute a panacea nor guarantee success, will probably be helpful to students:

1. Read the problem carefully, considering all the elements either expressed or implied. Assign letters or symbols to represent all elements which are unknown.
2. If possible, make a diagrammatic sketch representing the elements and the relationships in the problem.
3. Try to find some element which remains quantitatively unchanged throughout the problem, or for which two separate expressions can be obtained to make the basic equation.
4. Write this basic equation in words.
5. Translate the basic equation into algebraic language, using the data given or implied in the statement of the problem. It will probably be helpful in this connection to tabulate the pertinent data.
6. Solve the equation.
7. Check your solution *in the original problem.*
8. Be patient and painstaking. Do not become discouraged.
9. Use particular care to make your written work neat and orderly.
10. In problem situations that are not too involved, it is frequently desirable to make rough estimates of the answers before attempting to solve the problems.

Selection of Verbal Problems. It is generally felt that one serious limitation to the effectiveness of the work with verbal problems is that so many of the customary problems are outside the range of the students' ordinary experiences and they do not seem real to the students. It has often been said, and with entire truth, that many of the problems are artificial; that they lack reality because the students may feel no need for solving such problems by algebraic methods. The familiar "clock problems" are good examples of this. Why should anybody want to know the time between three and four o'clock when the hands of a clock will be together? And, if one *should* want to know this, could he not find it out more easily by the practical method of actually turning the hands of his watch to the desired position?

Or again, take such a problem as this:

Mary is half as old as Susan. In 8 years Mary will be $\frac{3}{4}$ as old as Susan. How old is each girl now?

Obviously, if this were an actual situation, one would have to know the ages of the two girls before he could set up the problem, and the problem could not conceivably have any personal interest or practical

importance to anyone except the acquaintances of the two girls, who probably would already know the ages of the girls.

Many teachers and writers are greatly concerned about this admitted lack of reality, genuineness, and practical importance in much of our verbal problem material. There is evidence, however, that this is not the sole criterion of student interest. As a matter of fact, many problems that are of the sheer puzzle type and without any practical importance whatever have been found to be highly interesting to students. On the other hand, some problems which contain elements of reality fail to elicit much student interest.[1]

Teachers should be primarily concerned with developing in their students a right attitude toward solving verbal problems. It is a mistake to try to give the impression that all verbal problems are practical or that the whole benefit to be derived from their study lies in their usefulness. There seems to be little justification for "dressing up" essentially unreal problems to give them a superficial semblance of reality.

> Such problems, if defensible at all, are defensible as mental gymnastics, and as appeals to the interest in mystery and puzzles. As such, they are better if freed from the pretence at reality. "I am thinking of a number. Half of it plus one-third of it exceed one-fourth of it by seven. What is the number?" is better than problems which falsely pretend to represent sane responses to real issues that life might offer.[2]

Breslich rates practicality and reality high in the criteria for suitable problem material but points out that a problem may be practical and still not suitable for elementary algebra.

> The situations of adult life involved in it may be too remote from the pupil's environment to be understood or appreciated. Problems taken from science and other studies may be ever so real without being real to the pupils who are to solve them. They may be beyond the mental reach of those students. The practical situations in a problem may be too complex for the beginner in algebra, and the facts involved may be unknown to him.[3]

Nevertheless, it will be generally conceded that, other things being

[1] See E. R. Breslich, "Problems in Teaching Secondary-school Mathematics" (Chicago: University of Chicago Press, 1931), pp. 183–187; also Jesse J. Powell, A Study of Problem Material in High School Algebra, *Contributions to Education* 405 (New York: Bureau of Publications, Teachers College, Columbia University, 1929).

[2] From E. L. Thorndike, "The Psychology of Algebra" (New York: The Macmillan Company, 1923), p. 138. By permission of The Macmillan Company, publishers.

[3] Breslich, *op. cit.*, p. 186.

equal, problems that lie within the experiences of the pupils and that exhibit genuine reality and practicality are more desirable than those which lack these characteristics. The criticism of much of the problem material on this score is a valid one. Teachers and textbook writers have not been oblivious to the shortcomings of the textbooks in this respect, and in recent years serious efforts have been made to improve this situation. Some progress has been made. However, problems which are genuinely real and practical, and which at the same time are within the comprehension and experience of young students, and which also submit themselves to elementary algebraic analysis appear to be very scarce. One of the greatest contributions that could be made to elementary algebra would be the compilation of an extensive list of verbal problems which would require solution by elementary algebraic processes and which would combine the elements of genuine reality, practical importance, and interest to the students. Such problems occur occasionally. Teachers should be always on the alert to detect, select, or create such problems to supplement those found in the textbooks.

Special Products and Factoring. The amount of time and effort generally spent in the study of special products and factoring in the ninth grade is probably in excess of that which can be justified. The uses of factoring are mainly confined to work with fractions and to the solution of certain equations, while the uses of special products are mainly in the direction of facilitating factoring. Since the equations and fractions suitable to ninth-grade work are relatively simple, the difficulty of the factoring and the special products studied in this grade should be in keeping with the difficulty and the requirements of these applications. The work in factoring should be confined to expressions involving a common factor, the difference of two squares, the square of a binomial, or the quadratic trinomial of the form $x^2 + px + q$. Consequently the work in special products need include only the product of a monomial by a binomial, the product of the sum and difference of two terms, the square of the sum or of the difference of two terms, and perhaps the product of two binomials of the form $(x \pm a)(x \pm b)$. Nothing is to be gained by including special products of the form $(ax \pm b)(cx \pm d)$, because such products can be found as quickly and with more assurance by direct multiplication, while the factoring of such products generally involves trial and should be checked by direct multiplication anyway.

So far as the special products themselves are concerned, the aim is to enable one to save time by writing the products down without going

through the details of multiplication. So far as their use in factoring is concerned, the aim is to suggest the factors and to enable one to write them down without the need of going through the details of trying and checking. The attainment of either of these aims is conditioned primarily upon the following considerations:

1. Knowledge of the type forms of the products to which the various special sets of factors (in type form) give rise
2. Knowledge of the type forms of the factors to which the various sets of special products (in type form) give rise
3. Ability to identify a particular pair of factors as belonging to a particular type and as giving a type product of a particular form
4. Ability to identify a particular product as belonging to a particular type and as having factors of a particular form
5. Ability to identify each element in a special given factor or product with the corresponding element in the type form to which it belongs, and to make the appropriate substitutions

Thus the two fundamental requisites are *knowledge of the type forms* of the various special kinds of factors and their associated products and *ability to recognize, identify, and associate* particular cases and particular elements with the general types to which they belong or with the corresponding elements therein. Consequently, the two foremost pedagogical questions are: how shall the student most effectively be brought to know these type forms, and how may he best become able to identify expressions in particular problems with the special types to which they belong?

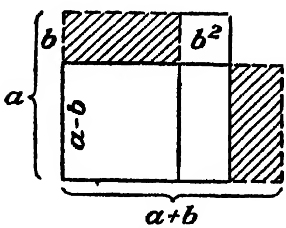

Fig. 23.

Geometrical and arithmetical illustrations will help to rationalize all these type forms and will provide a basis by which the student can easily reconstruct them. For example, consider the factorization of the difference of two squares:

$$a^2 - b^2 = (a + b)(a - b)$$

This geometric illustration shows clearly why the area represented by the product $(a + b)(a - b)$ is the same area as that represented by $a^2 - b^2$.

Now suppose a is 7 and b is 2. Then $a^2 - b^2 = 49 - 4 = 45$. But $(a + b) = 9$ and $(a - b) = 5$; and $(9)(5) = 45$. Thus we may also illustrate arithmetically the fact that $a^2 - b^2 = (a + b)(a - b)$. The other type forms may easily be illustrated in similar fashion.

To be of much use, the type forms must be learned thoroughly. They must be understood, and they must also be memorized. This means that the students must be shown in the beginning that the type forms either result from, or are verified by, actual multiplication. But it also means that in addition to this the students must have sufficient practice and drill upon these type forms to make the forms themselves become indelibly fixed in their minds.

This drill, however, should not be mere mechanical repetition. It should be made meaningful through constant illustration and application to specific examples involving both literal and numerical terms. The students should be kept conscious at all times that their principal job in connection with either writing special products or in factoring expressions is to *recognize* each problem as being a special case of one or another of the types which they have learned, to *identify* it with the type to which it belongs, and to make the appropriate substitutions of corresponding terms. Incidentally, this matter of identification and substitution of corresponding terms is one whose difficulties sometimes are not recognized by the teacher, although it actually constitutes the very crux of the difficulties which many students encounter. At no point is special attention more needed, and no point seems to be more neglected than this. There is no better way of ensuring real mastery of special products and factors than to give a well-organized special series of practice exercises on term-by-term identification and substitution.[1]

A topical study of special products and factors is a necessary but not a sufficient condition for permanent mastery. The applications which arise normally in connection with subsequent parts of the course are not in themselves sufficient to ensure continuation of satisfactory mastery even after it has been attained. If the skills are to be maintained, there should be a systematic program of subsequent drills spaced at suitable intervals throughout the remaining part of the course.

Exponents, Powers, and Roots of Numbers. Students will have little difficulty generally in performing operations involving exponents if they understand clearly the meaning of exponents. The fact that so many students make mistakes in these operations is doubtless due to the fact that the laws which govern the operations are too often developed hurriedly and without adequate care to ensure that the meanings are made clear. Results such as the following make it pain-

[1] The reader may well review at this point the discussion of the solution and evaluation of formulas in the preceding chapter.

fully clear that too often work with exponents comes to be a mere meaningless juggling of symbols:

$$a^3 + a^2 = a^5 \qquad (a^2)^3 = a^5 \qquad a^2(b^2) = (ab)^4$$

Such results would be less likely to occur if the concepts of positive integral exponents were adequately developed at the outset, for the laws of operation are simple and inevitable consequences of these concepts.

The meaning of a positive integral exponent is not hard to make clear. The approach is, in fact, suggested in most textbooks through one or two illustrative examples. It is pointed out, for example, that such a product as $(r)(r)(r)$ is written as r^3 for convenience and that the number symbol ^{3}, which is called the exponent in the expression r^3, merely indicates how many times r is to be used as a factor. Then two or three examples are given to illustrate the use of exponents in multiplication and division, the rules are stated, and exercises are given to afford the student opportunity for applying the rules.

The unfortunate thing is that this all seems so familiar and so obvious to the teacher that he is likely to assume that the one or two illustrations offered in the text are sufficient to make the matter equally obvious to the student, which, of course, is not generally the case. Examples of the foregoing type should be given until the students themselves are able to express the results in exponential form, tabulate them, and derive the rules for themselves. When they can do this, they will have the basis for a real understanding of positive integral exponents and of the laws for operating with such exponents. Also, in case they should ever become confused with regard to these rules, they will understand how to take the matter back to original meanings for their analysis of it and to rebuild the laws for themselves. They should, of course, have plenty of practice and drill in using the laws of exponents as such, after their meaning is understood. Above all, they should be trained to examine every problem carefully and to be sure that they understand precisely what is called for before proceeding with their work on it.

Since a positive integral exponent indicates *the number of times* the base is to be used as a factor, it is evident that special meanings will have to be attached to zero, negative, and fractional exponents. We want these meanings to be such that we may operate with these special kinds of exponents under the same laws as we use for positive integral exponents, so that the results of such operations will be consistent

in all respects with the results obtained by using positive integral exponents.

Consider the two laws for operating with positive integral exponents:
Law 1:

$$a^m \times a^n = a^{m+n}$$

Law 2:

$$a^m \div a^n = a^{m-n}$$

If the use of 0 as an exponent is to be defined so that these laws hold, then it is evident from Law 1 that

$$a^0 \times a^n = a^{0+n} = a^n$$

It immediately follows that $a^0 = 1$ is a justified meaning to give to the use of 0 as an exponent.

If a similar substitution is made in Law 2, we have

$$a^0 \div a^n = a^{0-n} = a^{-n}$$

But since $a^0 = 1$, this may be written

$$1 \div a^n = a^{-n}$$

Thus it immediately follows that a justified interpretation of a negative exponent is that it indicates the reciprocal of the same quantity raised to the corresponding positive exponent.

We cannot give meaning to fractional exponents until we define roots of numbers. The students should have gained from their arithmetic some understanding of the meaning of square roots and cube roots of numbers and of the symbols $\sqrt{}$ and $\sqrt[3]{}$. It is best, however, not to assume too much on this point. The square root of a number should be explicitly redefined as one of its two equal factors, the cube root as one of its three equal factors, etc., and the appropriate symbols carefully reassociated with their respective meanings.

As soon as this has been done, resort may be had again to analogy, as follows:

From the definition of square root, $(\sqrt{y})(\sqrt{y}) = y$. Also, if there were a number represented by the symbol $y^{1/2}$ and if it were subject to the laws of multiplication by exponents, then we should have $(y^{1/2})(y^{1/2}) = y^{(1/2+1/2)} = y^1 = y$. Now since $\sqrt{y}$ is one of the two equal factors whose product is y, and since $y^{1/2}$ is also one of two equal factors whose product is y, we may reasonably agree to think of $y^{1/2}$ and $\sqrt{y}$ as representing the same quantity. In other words, we may reasonably agree that $y^{1/2}$ shall have the same meaning as $\sqrt{y}$. Similarly,

the corresponding meanings may be attached to such expressions as $y^{1/3}$, $y^{1/4}$, $y^{1/5}$, etc.

Since $(x)(x) = x^2$ and also $(-x)(-x) = x^2$, it follows that

$$\sqrt{y} \cdot \sqrt{y} = y$$

and also $(-\sqrt{y})(-\sqrt{y}) = y$. It thus appears that every number has two distinct square roots, equal numerically but opposite in sign. The number $\sqrt{y}$ is called the "principal square root" of y. Since $\sqrt{y}$ has been identified with $y^{1/2}$, it follows that $y^{1/2}$ and $-(y^{1/2})$ are also symbols for the two square roots of y, the principal square root being represented by $y^{1/2}$.

Since these meanings have been arrived at by defining these fractional exponents in such a way that they will be subject to the laws of operation formulated for positive integral exponents, we may now apply those laws in such cases as $(y^{1/3})(y^{1/3})$. This obviously gives $y^{(1/3+1/3)}$ or $y^{2/3}$. It is equally clear that we have here $(y^{1/3})^2$. It is also clear that $(y^2)^{1/3}$ would give the same result, since $(y^2)^{1/3} = y^{2(1/3)} = y^{2/3}$.

By using numerous illustrative examples, carefully selected and arranged to bring out these analogies and identities, the students can be brought to understand what fractional exponents represent. It is well to have them tabulate the different forms of writing such expressions as

$y^{2/3}$	$(y^{1/3})^2$	$(\sqrt[3]{y})^2$	$(y^2)^{1/3}$	$\sqrt[3]{y^2}$
$x^{3/4}$	$(x^{1/4})^3$	$(\sqrt[4]{x})^3$	$(x^3)^{1/4}$	$\sqrt[4]{x^3}$
. . .	.	.		.
. . . . .		. .		
$x^{p/r}$	$(x^{1/r})^p$	$(\sqrt[r]{x})^p$	$(x^p)^{1/r}$	$\sqrt[r]{x^p}$
. . .		, etc ,	.	.

The careful comparison of the tabulated results of such equivalent forms will lead to a real understanding and generalization of the meaning of fractional exponents. In many cases the students themselves will be able to formulate the general relation that $x^{p/r}$ means the pth power of the rth root of x, or the rth root of the pth power of x. Numerical examples are helpful in making this meaning clear. A considerable amount of practice of the drill type should be given in order to fix the meanings and the identities of the equivalent forms firmly in the minds of the students.

Having now developed meanings and consistent general laws for exponents which may be either positive, negative, or zero, and either integral or fractional, it is clear that these laws may now be extended and made applicable to literal exponents as well, since these represent

numbers of one or another of the kinds just mentioned. While the students will have little occasion to use fractional or literal exponents in ninth-grade algebra, the teacher should have a clear concept of the organic and definitive nature of the progressive generalization of the laws to cover all kinds of exponents.

The Use of Parentheses. In connection with the use of parentheses the principal aims should be to have the students acquire (1) a clear understanding of the meaning of parentheses, (2) the ability to interpret correctly mathematical expressions in which parentheses are used, (3) the ability to employ parentheses correctly in situations where they are necessary or helpful, and (4) the ability to transform expressions involving parentheses into equivalent expressions which do not contain parentheses.

It is questionable whether a topical study of parentheses is either necessary or desirable, although most textbooks in ninth-grade algebra do offer such a treatment. As Breslich suggests,[1] much of the difficulty which students experience in the use of parentheses in algebra is probably due to the formal way in which the subject is presented to them. If they can be taught to think of parentheses simply as a symbol which indicates that the terms enclosed therein are to be regarded all together as one quantity, much, if not most, of the difficulty will disappear. Such a concept will enable the student to make his own analysis of every operation affecting or affected by the parenthetical quantity without the need of having special rules.

The most troublesome cases probably are those involving the "removal of parentheses" containing expressions of two or more terms. Special rules for the "removal" are generally given, often without adequate explanation. Hence students tend to look upon the removal of parentheses in such cases as a mechanical, rather than a rational, process. There follows a natural tendency to perform the operations with little or no thought of their meaning.

To correct this situation, it is suggested that the students' attention be focused upon the idea of "writing an equivalent expression without parentheses" rather than upon the "removal of the parentheses," and that the latter expression be completely discarded. Such a case as $3x - 4y + 11 - (2x + 3y - 5)$ will then become merely a problem in rewriting. It is apparent that each term must be rewritten with due regard to its sign, and, since the entire expression $(2x + 3y - 5)$ is to be subtracted from the foregoing terms, it is apparent that this means that each of its terms must be subtracted individually.

[1] Breslich, *op. cit.*, pp. 137–138.

Hence we write $3x - 4y + 11$ and then proceed to subtract, in turn, the terms $(2x)$, $(3y)$, and (-5). This procedure obviously gives

$$3x - 4y + 11 - (2x) - (3y) - (-5)$$

or $3x - 4y + 11 - 2x - 3y + 5$, which upon collecting terms gives $x - 7y + 16$ as the net result. Thus the expression has been rewritten without parentheses, and it has been accomplished without any special rule for the "removal" of the parentheses. The whole operation thus becomes one based upon understanding rather than merely upon the authority of an arbitrary rule.

It is probable that a freer and more systematic use of parentheses in such matters as the evaluation of formulas, the solution of literal and fractional equations, the addition of fractions, and various other algebraic operations would do much to prevent mistakes, to clarify the meaning, and to emphasize the usefulness of parentheses. The foregoing illustration is a case in point. Some mention of this has been made in certain sections of this and the preceding chapter, but for emphasis two additional illustrations will be given here.

1. Given the formula for the sides of a right triangle,

$$h^2 = a^2 + b^2,$$

let it be required to find the value of h in terms of a and b. Many students will thoughtlessly write "Since $h^2 = a^2 + b^2$, therefore $\sqrt{h^2} = \sqrt{a^2} + \sqrt{b^2}$ or $h = a + b$."

The square roots are taken of the separate terms of the right member of the equation rather than the square root of the right member as a whole. The fact that here $a^2 + b^2$ must be treated as a single quantity could easily be emphasized by writing the equation in the form $h^2 = (a^2 + b^2)$.

2. Given the formula for the volume of a right circular cylinder, $V = \pi r^2 h$, let it be required to find the volume of a particular cylinder using the values $\pi = 3.14$, $r = 1.7$, and $h = 5.5$. The following is not an exaggerated illustration of the careless way in which many students put down their written work for the evaluation of such a formula:

$$V = \overset{3.14}{\pi}\ \overset{1.7}{r^2}\ \overset{5.5}{h} = \ldots .$$

Such carelessness in writing is responsible for many arithmetical mistakes. If the students were trained to write out such a form as the following for the evaluation before substituting the numerical values,

there would be much less likelihood of mistakes occurring either in the substitution or the subsequent evaluation.

$$\begin{aligned} V &= \quad \pi \quad . \quad r^2 \quad . \quad h \\ &= (\quad)(\quad)^2(\quad) \\ &= (3.14)\,(1.7)^2(5.5) \\ &= \ldots\ldots\ldots\ldots \end{aligned}$$

Illustrative Tabular Analysis of a Teaching Problem.

RADICALS AND RADICAL EQUATIONS

I. What background may the students be expected to have with reference to this topic?

A. Use of radical sign and index to indicate square roots and cube roots of numbers

B. Meaning of literal numbers

C. Meaning and laws of positive integral exponents

D. Finding square roots and some cube roots, squares and cubes of numbers

II. What are the particular understandings and abilities which the student should acquire or strengthen through his study of this topic?

A. Things to know:

1. The precise meaning to be associated with a radical
2. The precise meaning of rational and irrational numbers
3. The precise meaning of similar radicals
4. The fact that in general only similar radicals may be combined by addition or subtraction
5. The precise conditions under which a radical is said to be in its simplest form
6. The fact that $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, and conversely*
7. The fact that $\sqrt[n]{a/b} = \sqrt[n]{a}/\sqrt[n]{b}$, and conversely*
8. The fact that in general $\sqrt[n]{a} \pm \sqrt[n]{b} \neq \sqrt[n]{a \pm b}$*
9. The principles which underlie the solution of radical equations
10. The meaning of extraneous roots and the necessity as well as the method of guarding against them in the solution of radical equations

B. Things to be able to do:

1. To combine similar radicals by addition or subtraction
2. To change radical expressions of the forms $\sqrt[n]{a \cdot b}$ and $\sqrt[n]{x/y}$ into the equivalent forms $\sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{x}/\sqrt[n]{y}$ (and conversely) with assurance and facility
3. To "simplify" radicals readily
4. To change radicals with different indices to similar radicals where possible so that they may be combined by addition or subtraction
5. To solve radical equations

* The exceptions to these properties need not be considered here.

6. To check the solution of radical equations and to detect and discard extraneous roots

III. What activities or procedures will enable the student most effectively to gain these desired understandings and abilities?

Each of the foregoing items should be subjected to a careful individual analysis before teaching. In general careful explanations by the teacher should be freely illustrated by numerical examples and accompanied by questions and discussion in which both the teacher and the students participate. Understanding should be followed by supervised practice or drill on the skills listed under II*B*. The drill should be specific, but with examples sufficiently varied to foster generalization and to give the students a feeling of assurance in identifying meanings and in recognizing appropriate procedures. Subsequent diagnostic testing and remedial instruction will probably be needed.

IV. What special difficulties may the students be expected to encounter in acquiring the desired understandings and skills?

A. Learning to simplify radicals such as $\sqrt{54}$ or $\sqrt{3x^3y}$

B. Learning to simplify fractional radicals such as $\sqrt{a/b}$

C. Combining radicals which are not given as similar radicals (*e.g.*, $\sqrt{27} + \sqrt{48}$). Instead of reducing these to similar radicals and then combining, some students are likely to say $\sqrt{27} + \sqrt{48} = \sqrt{75}$

D. Failing to check solutions of radical equations in the original equations

V. What specific suggestions, devices, and procedures will be most likely to help the student overcome these difficulties and to avoid these mistakes?

The particular shortcomings listed under IV give rise to mistakes which may be traced to inadequate understanding of the principles involved and to lack of experience in using these principles. The principles will doubtless have to be reexplained and discussed and the procedures reillustrated, perhaps several times. Each new discussion should aim to produce more complete and specific understanding and should be followed by specific and carefully supervised practice.

Considerable difficulty may be avoided if the students are taught to *write out in detail all the steps in the operations*, as follows:

$$\sqrt{54} = \sqrt{9 \cdot 6} = \sqrt{9} \cdot \sqrt{6} = 3\sqrt{6}$$
$$\sqrt{3x^3y} = \sqrt{x^2(3xy)} = \sqrt{x^2} \cdot \sqrt{3xy} = x\sqrt{3xy}$$
$$\sqrt{a/b} = \sqrt{ab/b^2} = \sqrt{ab}/\sqrt{b^2} = \sqrt{ab}/b$$
$$\sqrt{27} + \sqrt{48} = \sqrt{9 \cdot 3} + \sqrt{16 \cdot 3} = \sqrt{9} \cdot \sqrt{3} + \sqrt{16} \cdot \sqrt{3}$$
$$= 3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3}$$

Students often object to this detailed writing out of all the steps because they feel that it slows up their work. As a matter of fact, however, it has really the opposite effect by materially decreasing the number of mistakes. More important still, it is the best possible guarantee that the students will fully understand what they are doing and why they are doing it.

The foregoing section is illustrative of a pattern of analysis which can be made by any teacher with reference to any topic of algebra. The making of such analyses serves several useful purposes. It forces the teacher to a consideration of relative values in planning the various parts of his work. This enables him more effectively to disregard non-essentials and to build his teaching with appropriate emphases toward the attainment of the really significant outcomes. Again, it gives the teacher a more complete and organic view of the whole topic and enables him to see the various important elements in their relation to each other and to the whole. This makes for freedom, clarity, interest, and effectiveness in the developmental work. Finally, it calls his attention to the specific danger points which are, after all, the critical points in teaching. It forces the careful analysis of every individual concept and process to be taught, from the standpoints of anticipating inherent difficulties, tracing them to their specific causes, and then determining specific means for preventing or overcoming them. It will become increasingly valuable if it is revised from year to year in the light of students' mistakes in oral and written work.

No such analysis will ever be complete or perfect, because there will always be unpredictable factors. But every such analysis will be helpful in motivating the work and making it more efficient and effective through the judicious selection and organization of subject matter, the considered allocation of emphasis, and the selection of procedures and devices specifically designed to produce optimum understanding and mastery and to avoid, minimize, or correct specific misunderstandings or mistakes.

Exercises

1. What are the principal advantages of using the graphical method of solution in introducing the study of simultaneous linear equations in two unknowns? What are the relative disadvantages or limitations of this method compared to other methods?

2. In teaching simultaneous linear equations some teachers place more stress on the method of elimination by addition and subtraction than upon the method of substitution. Point out some disadvantages in neglecting to give some emphasis to the latter method.

3. Illustrate the solution of a system of two linear equations in two unknowns by the method of comparison.

4. "The length of a rectangle is 2 inches less than twice its width. If its perimeter is 20 inches, what are its dimensions?" Solve this problem by setting up one equation with one unknown. Then solve it by using two equations with two unknowns. Show that in the first case you really used two equations but solved one of them mentally.

5 Why is it that students find it more difficult to operate with algebraic fractions than with arithmetical fractions? Be explicit.

6. Make a list of the specific difficulties which you would expect students to encounter in learning to reduce fractions Prepare explanations designed to help students avoid or overcome these difficulties.

7. Do the same with reference to the addition of fractions.

8. Point out the potential dangers of using the term "cancellation" in first-year algebra. Also point out basic considerations essential to a clear understanding of the true nature of the process.

9. In dividing one fraction by another we generally use the rule, "Invert the divisor and then multiply." If a student should ask you why that rule is valid, what explanation would you give him?

10. In explaining to ninth graders how to perform the operation $\frac{6n}{5} \div 3$ would you teach them to divide the numerator by 3, or would you teach them to regard the divisor as a fraction $\frac{3}{1}$ and then to use the customary rule for dividing by a fraction? What advantages can you see in the method you prefer?

11. If one of your students should write $\frac{a}{x} + \frac{5}{x+3} = \frac{a+5}{2x+3}$ just what would you do to get him to see his error, and to understand why it is an error, how to correct it, and how to check his result?

12. From ninth-grade textbooks select 10 examples in the addition of fractions which, because of their complexity or difficulty, you consider inappropriate for average ninth-grade students

13. Build up a set of 50 practice exercises in the addition of fractions suitable for first-year algebra, starting with extremely simple ones and shading gradually into more and more complicated ones Never introduce more than one new complication at a time

14 Summarize in a brief list of categorical statements the considerations which lie at the foundation of the rule for the algebraic addition of fractions.

15. In adding fractions it is necessary to preserve the original value of each of the fractions involved. Is this necessary in solving equations containing fractions? Explain and illustrate.

16. Give a clear and connected résumé of the discussion of teaching the solution of equations containing fractions, as it is presented in this chapter. Emphasize the salient points.

17. How do you account for the fact that students generally have more trouble with verbal problems than with most other parts of algebra?

18. Enumerate any difficulties which you think students may have in solving verbal problems. Be as specific as you can.

19. Construct a diagnostic test which would help you locate and identify the

types or sources of difficulty which you listed in the preceding exercise. Explain just how you would use the results of this test.

20. Read carefully 50 verbal problems chosen at random from a ninth-grade algebra textbook, and list any words which you think the students might not understand clearly. Do you think the vocabulary difficulties would be a serious source of trouble with verbal problems?

21. Verbal problems are often first presented and taught under various "types" such as coin problems, mixture problems, etc. Compare the relative advantages and disadvantages of this practice.

22. Enumerate some specific improvable abilities or skills which are involved in solving verbal problems.

23. The learning of many definitions and rules verbatim used to be heavily stressed in algebra. State and justify the present prevailing opinion with regard to the learning of definitions and rules.

24. Explain in detail how you would make it clear and convincing to a class that the following definitive relations are consistent with the laws of operation with positive integral exponents:

$$x^{1/3} = \sqrt[3]{x} \qquad x^{-3} = \frac{1}{x^3} \qquad x^0 = 1$$

25. Does your explanation constitute a rigorous proof of the relations set forth in the preceding exercise, or does it merely *define* fractional, zero, and negative exponents? Explain.

26. What are the reasons for teaching special products and factoring, and why should they be taught together?

27. Read carefully the chapter or section dealing with special products and factors in a ninth-grade algebra textbook, and make a list of all the words which you think a good many of the students might not understand clearly.

28. What special products and what types of factoring are suitable for ninth-grade algebra? Examine some textbooks, and see whether or not you find any which in your opinion are not suitable for the ninth grade.

29. Discuss the particular knowledge and abilities which, as set forth in this chapter, are regarded as necessary to adequate mastery of special products and factoring.

30. What do you think is the best way to get students to understand the meaning and use of parentheses? How important is this? Why?

31. Write a critical review of one textbook in first-year algebra.

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CHAPTER XIV

THE TEACHING OF ALGEBRA IN THE SENIOR HIGH SCHOOL AND THE JUNIOR COLLEGE

The subject matter of algebra is very cumulative; hence at the higher levels it makes continual use of the concepts, principles, and operations developed in the ninth-grade work at the same time that new concepts and new principles are introduced and studied. From one standpoint the broader objectives of the instruction at this more advanced level offer certain points of contrast with the objectives of the junior high school. In particular, the elective status of the higher courses implies a somewhat different personnel in such classes. These students are likely to be inherently more interested and capable than typical junior-high-school students. It is also quite probable that they are studying algebra because of its subsequent usefulness in academic or professional fields. In view of these facts, the technical aspects of algebra may legitimately come to occupy a relatively more important place among the instructional aims. This, of course, does not imply any lessening of the emphasis upon understanding, but it does imply a progressively increasing insistence upon the mastery of the algebraic tools.

Qualitatively, the general aims are the same as before, *viz.*, to develop and clarify understandings, to produce familiarity with the terminology, notation, and symbolism of algebra, and to perfect operational facility. The expanding field introduces new difficulties for the students which are offset only in part by their added maturity and by the operation of the principle of progressive selectivity. Intelligence and reasonable mastery of prerequisite work are necessary conditions for the successful study of algebra at these higher levels, but they are not in themselves sufficient. The teacher's role is still one of extreme importance, and his effectiveness will depend in great measure upon his ability to anticipate and overcome specific difficulties which the students are likely to have in connection with particular topics.

Review Work in Beginning Intermediate Algebra. In beginning the course in intermediate algebra, teachers generally find it necessary to make some review of the work of the ninth grade. Indeed, most

textbooks in the more advanced algebra courses begin with several chapters which are essentially reviews of various phases of the earlier work.

Much time may be wasted in this review work, however, unless it is carefully and purposefully planned. It is well at the outset to take a rather careful inventory of the algebraic equipment of the class. This may be done by administering a comprehensive inventory test. If this is done at the beginning and if the results are immediately tabulated and analyzed, it is possible for the teacher to have very soon a fairly accurate picture of the needs of the class for review work. If these tabulated results are used as a basis for planning and conducting the review, it will be possible to concentrate attention on those places where the need is evident and to dispense with unnecessary work on other topics.

There is an advantage in spreading the review over some 2 or 3 weeks and giving it in small doses interspersed with new work instead of concentrating it all in the first few days and leaving it as finished business. If the students get nothing but review work for several days, it becomes tiresome. Some new work offered along with the review provides both variety and incentive. The distribution of time in this way also makes for more effective learning. Among the things which are likely to need special attention in this review work may be mentioned the language of algebra (substitution and evaluation), positive and negative numbers, easy formal work in the fundamental operations, the solution of literal equations, and easy work with fractions.

Even Simple Details Need to Be Taught Specifically. Teachers often overlook the very important fact that much of the difficulty which students have with algebra is due to their failure to understand and generalize certain critical *details* of the work or to appreciate the bearing of these details upon the work as a whole It is not at all uncommon for students to find themselves confused and their progress blocked merely because of lack of understanding of some apparently minor point. In many cases, when the proper cue is given or the troublesome detail cleared up, the student is able to proceed with independence and satisfaction. Many such details appear so familiar, simple, and obvious that teachers do not recognize them as trouble-makers, but it is not wise to assume too much in this respect, even in the matter of recognizing equivalent expressions and in making proper substitution. The following examples will make this clear.

Frequently students experience difficulty because they fail to *recognize* the equivalence or identity of certain algebraic expressions. This

is particularly true with reference to indicated products and quotients, roots, and exponents. For example, the expressions $x^3/2$ and $\frac{1}{2}x^3$ are precisely equivalent, but even in such simple cases it is not uncommon to find students who are unable to recognize this fundamental equivalence. Other examples of equivalent expressions in which the identity is not always readily discerned are found in such illustrations as the following:

$$\frac{2}{3}xy,\ \frac{2x}{3}(y),\ 2y\left(\frac{x}{3}\right),\ \text{and}\ 2\left(\frac{xy}{3}\right)$$

$$\frac{1}{2}h(b+B),\ \frac{h}{2}(b+B),\ \left(\frac{b+B}{2}\right)h,\ \text{and}\ \frac{(b+B)h}{2}$$

$$\sqrt{7x}\ \text{and}\ \sqrt{7}\cdot\sqrt{x}$$

$$\frac{\sqrt{x}}{\sqrt{y}}\ \text{and}\ \sqrt{\frac{x}{y}}$$

$$\left(\frac{x}{y}\right)^3\ \text{and}\ \frac{x^3}{y^3}$$

$$x^3y^3\ \text{and}\ (xy)^3$$

Students also frequently fail to recognize the fact that, whereas in the multiplication of a polynomial the multiplier must be applied to each term in the polynomial, in the multiplication of an indicated product of two or more factors the multiplier is applied to only one factor (any factor) of the indicated product. Likewise, they do not always recognize that to multiply a fraction means to multiply only its numerator. Hence such erroneous statements as

$$r(x+y)(a+b) = (rx+ry)(ra+rb)$$

or

$$3(a/b) = 3a/3b$$

are of common occurrence.

Another case in point is found in connection with the substitution of one expression for another in a function or a formula. For example, the teacher immediately sees in the expression $8x^6 - 125y^9$ the difference of two cubes and associates the expression at once with the form $a^3 - b^3$. It seems so natural to identify $2x^2$ with a and $5y^3$ with b that it may not even occur to him that the student will have any difficulty. However, it takes specific teaching, deliberately designed to emphasize this particular association, to give the students a feeling of security and assurance in handling such situations. The acts of recognition, identification, association, and substitution which are

involved are things which will be successfully mastered in most cases only if they are taught specifically and if considerable special practice is given in performing them.

Many other similar illustrations could be given. There is nothing in these situations themselves which is inherently difficult. The correct interpretations can invariably be explained easily and to the entire satisfaction of the students by dealing with them deliberately and specifically. Numerical examples are often sufficient, but the students must be continually impressed with the need for associating the specific illustration with the generalized procedure. The trouble is that teachers do not keep themselves sensitive to the fact that the students need to be taught to generalize these details. They take too much for granted. The reason why most students do not react properly to such situations is that they are not taught to do so. The teacher, if he gives any thought to it at all, generally assumes that the principles or the generalizations are so obvious that they need no discussion. It is a fact, however, that this assumption cannot safely be made. The only reasonable assurance of the effective functioning of these recognitions, identifications, and generalizations is to give special instruction and carefully planned practice designed specifically to this end.

Solving Quadratic Equations in One Unknown. Five methods for solving quadratic equations are commonly taught. They are (1) solving by the graphical method; (2) solving by inspection (in the case of incomplete quadratics); (3) solving by factoring; (4) solving by completing the square; and (5) solving by use of the quadratic formula.

The graphical method is not strictly an algebraic method and can give only approximate solutions, but it is very valuable in clarifying the meaning and nature of the two roots, just as the graph itself is useful in bringing out certain important characteristics of the function. There are four essential things for the students to understand: (1) that the graph is the geometric picture of a *function* of the independent variable and that, as the independent variable takes on different values, the function also takes on different values; (2) that, since the general form of the quadratic equation is $ax^2 + bx + c = 0$, we seek to find those values of x which will make the function $y = ax^2 + bx + c$ have the value zero; (3) that the only points on the graph for which the function can have the value *zero* are those points which are on the *x-axis;* and (4) that consequently the abscissas of those points give the values of the independent variable which satisfy the equation. These values are sometimes called the zeros of the function. The graph also can be used effectively to show why two real and distinct

roots may exist, why the roots are sometimes real and equal, and why, in certain cases, there are no real roots.

Because it is slow and tedious, the graphical solution should not be required for many problems. Its purpose will have been served when it has been used in a few cases to clarify and amplify the students' understanding of quadratic equations and their solutions. The major part of the work with quadratic equations should be concerned with algebraic rather than graphical solutions.

The second method mentioned above—that of solving by inspection—is so simple and obvious that often it is not even listed as a method of solution. The third offers little difficulty to students except possibly in the factoring, and this will merely require careful attention. There is nothing inherently new about the process, but there is one point that may need reexplanation. When an equation such as $x^2 + 5x - 14 = 0$ is given in factored form as

$$(x + 7)(x - 2) = 0,$$

it is not always clear to the students why one has the right to set the factors separately equal to zero and thus to get two linear equations. The justification for this should be made clear. That is, it should be explained that, if either of the factors is zero, the function itself will be equal to zero and the equation is thus satisfied; conversely, that the *only* way in which the equation can be satisfied is for at least one of the factors to be equal to zero.

In taking up the method of solving quadratics by completing the square, it will be well to review briefly the characteristics of the perfect trinomial square of the form $x^2 + 2bx + b^2$ in order that the students may get well in mind the relation between the coefficient of x and the term free of x. This is the crux of the whole matter, and it should be carefully explained and freely illustrated. It is desirable that the students themselves should make the generalization and be able to state it in words. The generality of this method of solution should be emphasized and contrasted with the lack of generality (in terms of real numbers) of the factoring method. It will not be necessary, however, to spend much time in actually solving equations by this method, since its principal function here is to provide a means for developing the general quadratic formula.

The general quadratic formula is of such special importance and usefulness that it should be thoroughly mastered by every student. Its development requires the use of the method of completing the square and provides an excellent review of operations with literal num-

bers. The development should be carefully explained and the students tested for their understanding of every step. The formula itself is indispensable, and every student should memorize it and use it until he is perfectly familiar with its form and meaning.

In connection with the study of this formula, the teacher should see to it that the students understand the meaning of the discriminant and that they understand why it is possible, in a quadratic equation under proper hypotheses of reality or rationality of the coefficients, to determine the nature of the roots from a study of the discriminant alone, without solving the equation. To this end it is desirable for the teacher and students to examine and discuss together a variety of quadratic equations with numerical coefficients, after which the students may well be given exercises such as the following.

Exercise. In each of the following quadratic equations the coefficients are rational numbers Indicate in each case the nature of the roots so far as this can be determined from the discriminant alone. Do not solve the equations.

Equation	Discriminant $b^2 - 4ac$	Nature of roots		
		Real or imaginary	Equal or unequal	Rational or irrational
$x^2 + 7x + 6 = 0$				
$3y^2 + 7y + 2 = 0$				
$2x^2 - 2x + 11 = 0$				
$x^2 - 10x + 25 = 0$				
$5x^2 + x - 7 = 0$				

Exercises of this sort demand and develop insight into the role of the discriminant in determining the nature of the roots and contribute materially to the understanding and appreciation of the generality of the formula. In this way they form an excellent point of departure for subsequent study of the theory of equations.

Not infrequently students have difficulty in dealing with equations such as $x^6 + 13x^3 + 36 = 0$. The difficulty almost invariably lies in the failure to recognize that, while this, for example, is a sixth-degree equation in x, it may be regarded as a quadratic equation in x^3 and so may be solved easily for x^3, the values of x itself then being readily found by taking the cube roots of x^3. To help the students become

sensitive to the possibility of reducing such equations to quadratic form, a variety of examples should be given, the quadratic form being written out in each case. Thus $x^8 + 5x^4 - 2 = 0$ may be written $(x^4)^2 + 5(x^4) - 2 = 0$, or, if preferred, a different letter, say z, may be substituted for x^4 so that the equation would become $z^2 + 5z - 2 = 0$. The illustrations should include examples involving radicals and fractional exponents, such as $x + 3\sqrt{x} - 18 = 0$ and $2x^{2/5} + 8x^{1/5} + 12 = 0$. After a number of illustrative examples have been given, the principle may be generalized to help in subsequent recognition of such cases.

In the work of the twelfth grade or of the first year of college algebra there should be extensive applications of the foregoing methods to the solution of quadratic equations in one unknown. This work would necessarily involve the use of radicals and imaginary and complex numbers. The quadratic formula as a general solution should be stressed. Such work may appropriately be extended to include the investigation of certain other general properties of quadratic equations, particularly the relations existing among the roots and the coefficients. It should lead to the subsequent study of quadratic equations and systems of equations in two unknowns.

Many students who are able to apply the quadratic formula explicitly in determining the roots of a given equation find themselves at a loss when confronted with situations in which the formula is implicitly involved. The following illustrative examples are cases in point.

In each of the following equations determine the real values of k for which the roots will be equal:

$$4x^2 - 12x + k = 0$$
$$2kx^2 + 5x + 1 = 0$$
$$x^2 - 8kx + 4 = 0$$

There usually will be a few students who will be able to sense for themselves the role of the discriminant in such cases, but for many of them this will need to be pointed out and illustrated specifically. In particular it will be necessary to review the fact that the equality or inequality of the roots of such equations is determined solely according to whether the discriminant $b^2 - 4ac$ is or is not equal to zero, and consequently the condition for their equality is that k must be of such value as will make this discriminant zero. That is, the student must come to sense the fact that, in order to produce the required condition and to discover the required value of k, the discriminant of the particular equation must be set equal to zero and the resulting equation solved for k.

The relations between the roots and the coefficients should be carefully developed. Most textbooks give the bare symbolic development of these formulas but are usually lacking in explanatory comment. To supply adequate explanation of the development and pointed comment with reference to applying these relations must be the task of the teacher. He must use a variety of problems which will provide the student with the opportunity of seeing the formulas applied both explicitly and implicitly. For example:

Find: The sum and product of the roots without solving.

$$5x^2 - 3x + 8 = 0$$
$$2x + 5 = x^2$$

Given: The equation and one root, as indicated; find the other root without solving.

$$x^2 - 11x + 24 = 0 \qquad \text{(one root is 8)}$$
$$2x^2 - 17x + 33 = 0 \qquad \text{(one root is 3)}$$

Given: The roots as indicated: write the equations.

Roots are $5/2$ and -6; equation: __________

Roots are k and k/a; equation: __________

Other and varied examples will be found in any text in college algebra. Those which are to be used for illustrative purposes or for practice work should be carefully selected by the teacher. The main criterion should be the extent to which the exercise lends itself to clarifying and emphasizing the particular point in question.

Systems Involving Quadratic Equations in Two Unknowns. The students' previous experience in solving simultaneous equations may be assumed to have been confined to systems of linear equations. In teaching the solution of such systems of equations, many teachers give preponderant emphasis to the method of elimination by addition or subtraction. The disadvantage of failing to give due emphasis to the method of substitution now becomes apparent, because this method is generally applicable to the solution of systems involving quadratics while the method of addition and subtraction is not applicable in all cases. Therefore it may be advisable, in taking up the study of this topic, to give a *brief* review of the solution of systems of linear equations by substitution. The students should be made sensitive to the importance of the method as a general algebraic method for solving quadratic systems in two unknowns.

Systems of equations involving quadratics fall into two general classes: (1) systems containing one linear and one quadratic equation

and (2) systems in which both equations are quadratics. The latter class may be further divided to advantage into homogeneous and non-homogeneous systems. The special methods and devices used in the solution of these systems vary according to the forms of the equations involved in the particular systems under consideration. Teachers should make every effort to see that students do not use these various devices blindly but are trained to look for the reasons why the devices work. In doing this they will be acquiring the ability to analyze systems of equations and to determine for themselves what procedures will be most likely to yield the desired solutions. It is highly important that the teachers use discretion and care in the selection of problems to be assigned.

Textbooks generally present these devices with appropriate illustrations, but often without emphasizing either the peculiar characteristics of the forms to which the various methods are especially adapted or the *reasons why* a particular method is especially suitable for handling a particular form. It must be the teacher's primary task to see that the students are given insight into the *reasons* underlying the use of the various devices in particular situations as well as a knowledge of the devices themselves. Otherwise rote work is inevitable.

Useful and economical as these special procedures are, the student should not be allowed to forget that as a rule they are merely means for reducing the tedious labor often involved in the more general method of solving one equation for one variable in terms of the other and then substituting the result in the other equation.

The graphical method of solving quadratic systems is a laborious method and, of course, gives only approximate solutions, but it probably gives the average student more real intuitive insight into the nature of the solutions than any of the strictly algebraic methods of attack. In particular, it is useful in explaining why a system containing two quadratic equations has four solutions (real or imaginary) and why a system containing one quadratic equation and one linear equation has only two solutions.

The set of graphs on page 383 includes representation of all types possible in quadratic systems. A thorough discussion and comparison of such graphs as these and their associated equations should give the students an intuitive basis, which they too often fail to get, for interpreting the solutions of such systems. Moreover, the use of graphs in this connection will give the student an initial acquaintance with some of the basic concepts of analytic geometry. In this way it serves, not only to clarify his present work in algebra, but also to stimulate

his interest in, and to provide a basis for, a thorough understanding of the analytic geometry that lies ahead.

Ratio, Proportion, and Variation. Most students have a fairly clear understanding of the meaning of ratio and proportion by the time they reach the eleventh grade. The concepts of ratio and proportion are used, at least intuitively and by implication, even in the arithmetic of the elementary school and to a greater extent in the subsequent

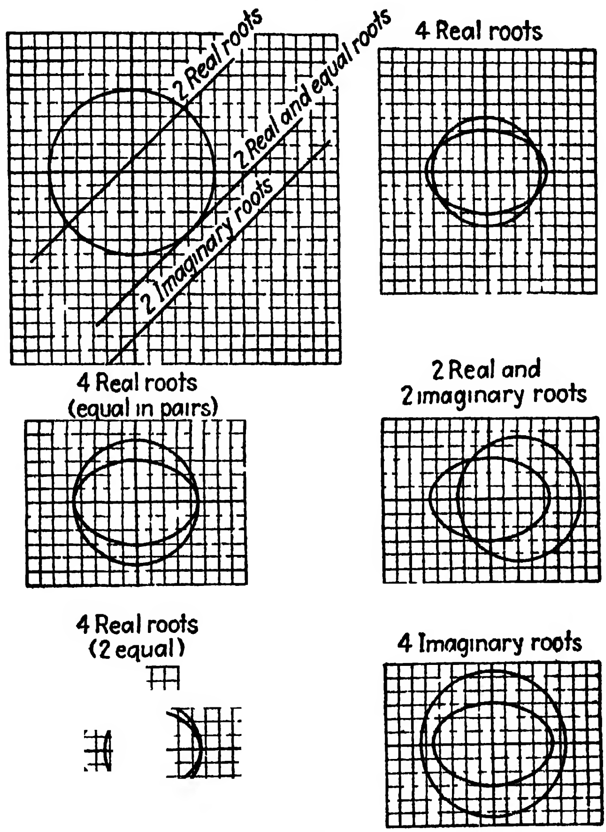

Fig 24.

work of the junior high school Most textbooks in ninth-grade algebra or general mathematics contain some systematic, though elementary, treatment of this subject, and a more extensive treatment is found in the study of similar geometric figures in intuitive and demonstrative geometry and in numerical trigonometry. Consequently students in the advanced courses may be expected to have some familiarity with the concepts and techniques of ratio and proportion.

The concept of variation also will be familiar to them, but with

intuitive rather than analytical implications. In all probability the form and meaning of the symbolic representation will be new. However, since it is often more convenient and effective to use the variation form of notation than to use the proportion form, the systematic analytic study of variation should undoubtedly find a place in the advanced work in algebra. It is not a particularly difficult topic to teach, but it needs to be taught thoroughly. Such explanation as may be given in the textbook will necessarily be condensed and will need to be materially supplemented by the teacher.

The teacher should have in mind two specific objectives: (1) to see that the students get a clear understanding of what is meant by direct variation, inverse variation, and joint variation; and (2) to see that they learn how to set up these relations in the form of equations involving a constant of variation and to see that they understand why these equations necessarily represent the different types of variation. A variety of illustrative formulas, graphs, and problems can be used effectively to bring out the meaning of the equations $y = kx$, $y = k/x$, $x = kyz$, $y = kx^2$, $y = kx^2/z^3$, etc. Innumerable examples involving variation may be drawn from arithmetic, geometry, physics, and other sources. The following illustrative examples are designed to give the students training in translating verbal statements of relationships into equations of these types. In each case the student should be asked to translate the statement into an equation involving a constant k and to indicate the numerical value of k where he can.

1. The perimeter of a square varies directly as the side.
2. The area of a square varies directly as the square of a side.
3. The circumference of a circle varies directly as the radius
4. The volume of a sphere varies directly as the cube of the radius.
5. The base of a rectangle of constant area varies inversely as the altitude.
6. The strength of an electric current (amperage) varies directly as the voltage and inversely as the resistance.
7. The amount of simple interest varies jointly as the principal, rate, and time.
8. The volume of a confined gas varies directly as the absolute temperature and inversely as the pressure.
9. The intensity of illumination from a given source varies inversely as the square of the distance from that source.

Many applications should be given in the form of problems in order that the students may become thoroughly familiar with these forms and may acquire the ability to apply them correctly in appropriate situations and to use them with proficiency and assurance. The students will realize, of course, that there are many situations involving variation in which the relationships are so complex that the mathemati-

cal laws governing them are not known. They should be impressed, however, with the fact that, whenever we can set up an analytical representation of the mathematical law governing such variation, we are thereby providing ourselves with a powerful instrument for studying and predicting the behavior of the variables.

The study of variation affords an excellent opportunity to clarify and emphasize the nature of functional relationship and of functional thinking. In fact, the concept of function is inseparable from the concept of variation, and the two should be stressed together, both concepts being abstracted and clarified through the interweaving of graphic illustrations and numerical evaluation of formulas or polynomial functions. The concept of continuity should be emphasized in connection with the graphs of such functions. This concept should be associated with the concept of the *changing values* of the variables. These, in turn, are to be associated with the shifting coordinates of a point moving at will on the graph; and the graph itself should be associated, through these notions, with the formula or function which it represents. Proper attention to these concepts and associations of ideas will go far toward developing a real idea of the nature of variation at the same time that it gives clear emphasis to the general notion of dependence and functionality.

Arithmetic and Geometric Progressions. The first fundamental necessity in teaching progressions is to see that the students become really sensitive to the way in which the progressions are built up, *i.e.*, the precise way in which each term after the first one is obtained from the preceding term. There are two approaches to this problem. Textbooks generally start by giving in each case a definition of the type of progression being discussed, calling attention to its distinguishing characteristic (the common difference between two consecutive terms or the common ratio of the one to the other), giving a few numerical illustrations, presenting in symbolic form the first few terms, and proceeding then to the development of the formula for the nth term. In this procedure the definition and the concept forms the starting point, and the numerical examples merely illustrate, supplement, and enrich the concept.

Some teachers feel, however, that better results are to be secured by more or less reversing this order of things. Under this plan various simple numerical illustrations of progressions are presented at the beginning without any definite laws or conditions governing the relations among the terms being given. The students are merely asked to try to find the way in which each series is built up or to discover

for themselves the characteristic relation governing each of the series. This approach obviously stresses discovery rather than definition as a starting point. After the student has discovered the relations governing several progressions, he is asked to formulate a statement expressing this relation for each type of progression and finally to express these laws or relationships symbolically as formulas for the *n*th term of the series. It is held that many students can do this if the illustrative series are simple and appropriately chosen.

Advocates of this approach contend that students who discover the relationships for themselves are likely to understand the characteristic laws better, remember them more vividly, and apply them more readily than those who start with ready-made definitions and formulas. There are probably some grounds for this contention. There is no conclusive evidence, however, that either of these approaches is markedly superior to the other. In either case numerical illustrations are indispensable, and in either case it will probably be necessary for the teacher to help many of the students in setting up the symbolic formula for the typical *n*th term in the particular type of progression being considered.

Derivation of the formulas for the sum of the first *n* terms is always interesting to the students, not only because of the importance of the formulas themselves, but also because of the clever special devices by which the formulas are obtained. These devices provide an excellent means for stimulating interest and should be capitalized.

The most serious difficulty which most students encounter in the study of progressions is in knowing how to go about the insertion of a given number of arithmetic or geometric means between two given numbers. This difficulty can easily be dispelled, however, if the students can be made sensitive to the fact that all they need to do is to find the common difference or the common ratio and that they can do this in either case by taking the formula for the last term and solving it for the common difference or the common ratio in terms of the other elements of the formula. For several reasons it is better for them to carry out these solutions as they are needed rather than to memorize the formulas for *d* and *r*. Besides reducing the amount of memorizing, it enhances the students' understanding of the formulas and provides excellent practice in the solution of literal equations.

Arithmetic and geometric progressions have extremely interesting and important applications which should be pointed out to the students. In any linear function, for consecutive integral values (in terms of any real unit) of the independent variable, the corresponding values of the function form an arithmetic progression, and conversely. This

may be illustrated both graphically and numerically. Doubtless the most important general application of the geometric progression is the compound-interest law. Here the teacher has the opportunity to emphasize the usefulness of geometric progressions by showing that this law is applied, not only to the computation of compound interest, but also to important problems in various fields. The following examples are illustrative:

Chemistry: problems associated with the disintegration of radioactive substances
Physics: the adiabatic law for gases; rates of cooling
Biology: problems associated with the growth of colonies of bacteria and abnormal tissue growth
Economics: problems of investment, insurance, debt funding, and installment buying
Sociology: problems associated with population growth

The study of harmonic progressions is a topic which has important but highly specialized application, and it is usually taught (or retaught) in connection with these applications. It is of less general interest than arithmetic and geometric progressions and is often omitted even from courses in college algebra. The same may be said with reference to the convergence of certain infinite geometric series. Because of the specialized application of these topics, no discussion of them will be given in the present chapter.

Complex Numbers. In introducing students to the study of imaginary and complex numbers it is well to review the previous steps in the extension of their number concepts from positive integers to common and decimal fractions and later to irrational numbers and negative numbers. They will be interested in realizing that each of these extensions was made in response to a need; that, when situations were encountered which could not be interpreted or explained adequately by use of positive integers alone, fractions were invented to do the job; and similarly for negative numbers. The point should be stressed that these new kinds of number have been sheer *inventions* made to serve a purpose and that they take their meanings from definition. This having been established, the students will tend to be in a receptive frame of mind for further extension of their number ideas.

The meaning of an imaginary number should be made clear at the outset. There should be no mystery about it, and there need be none. An explanation along the following lines will make clear the meaning which is to be attached to such numbers and will dispel much of the intellectual reservation and even antagonism which often exists with reference to this radically new concept.

Let us consider some negative number, say -9. Can we find its square root? The $\sqrt{-9}$ cannot be $+3$ because $(+3)(+3) = +9$, nor can the square root be -3 because $(-3)(-3) = +9$. In fact the square root of -9 cannot be any positive number and it cannot be any negative number, because the square of either a positive or a negative number is positive. What then can it be? The only kind of numbers we know about up to now are positive and negative numbers. Since the square root of -9 cannot be either of these, we must *invent* another kind of number which we shall *define* as being the square root of a negative number and which we shall *call* an *imaginary number*. This is, in fact, what mathematicians have done. They have recalled that the square of the square root of any number gives that number (*i.e.*, $\sqrt{7} \cdot \sqrt{7} = 7$; $\sqrt{23} \cdot \sqrt{23} = 23$; etc.), and they have said that, since this is the meaning of a square root, therefore it must be true that $\sqrt{-9} \cdot \sqrt{-9}$ must be equal to -9. Since they call both positive and negative numbers *real* numbers and since $\sqrt{-9}$ cannot be a real number, they call it an *imaginary* number. Other imaginary numbers are $\sqrt{-48}$, $\sqrt{-2}$, $\sqrt{-1}$, $\sqrt{-364}$, etc. The special symbol **i** is used to denote the imaginary number $\sqrt{-1}$.

This explanation will, of course, need to be amplified by the teacher, but it indicates the main avenue along which the students' thinking should be directed. It should be made clear that, in thus defining imaginary numbers, we make them subject to all the normal laws of operation which we use with real numbers. It is very important that students understand this clearly, and this understanding is facilitated, and confusion avoided, by the consistent use of the symbol **i** in place of the radical $\sqrt{-1}$. Thus there is no difference between applying the laws of exponents, the law of signs in multiplication, etc., to numbers expressed in terms of **i** and applying these laws to numbers expressed in terms of x or y or a or any other literal symbol. The peculiar cyclic nature of the successive integral powers of **i** often makes it possible to simplify the results of such operations, but this characteristic itself is a direct consequence of the laws of exponents applied in the usual manner to the number **i**, and of the definition of that number.

The principal points at which difficulty may be anticipated are: the establishment of the definition and meaning of imaginary numbers; the definition of the symbol **i**; the firm fixation of the principle that *operations* with numbers in terms of **i** are carried on in exactly the same fashion as operations with numbers expressed in terms of any other letter; and the establishment of the successive positive integral powers of **i** which give a recurring series of numbers $+i$, -1, $-i$, $+1$.

When these properties of imaginary numbers have been well established in the minds of the students, the subsequent definition of, and work with, complex numbers should present little difficulty. Care must be taken in defining equal complex numbers. Conjugate complex numbers must also be defined. The principle of combining real parts and imaginary parts in the addition or subtraction of complex numbers can be readily explained by the fundamental law that only like terms can be combined. The real parts themselves are obviously like terms, as are the imaginary parts, while the real and imaginary parts are, just as obviously, unlike terms.

The generality of complex numbers, *i.e.*, the fact that a complex number may be either a real number, a pure imaginary, or a combination of these, is intriguing to most students when they come to understand and appreciate it. This generality may be made clear by considering that in the general form of the complex number $a + bi$ the a and b are any real numbers. Therefore if $a = 0$ the complex number becomes $0 + bi$ or simply bi, a pure imaginary. On the other hand, if $b = 0$ the complex number becomes $a + 0i$ or simply a, a real number.

The geometrical representation and treatment of complex numbers is extremely interesting to students and helps to make their concepts of these numbers much more tangible. This work is not extremely difficult and may well be introduced even in the eleventh grade, at least to the extent of making the students familiar with the method of representing complex numbers graphically and with the basic principles of simple vector addition. For twelfth-grade or college students who have had trigonometry, this work may well be extended to include the representation of complex numbers in trigonometric form, multiplication of two numbers in polar form, DeMoivre's Theorem, finding the nth roots of a complex number, and the division of complex numbers expressed in polar form.

Other Topics in Algebra. In a volume such as this it is impossible to consider in detail the teaching procedures connected with all the topics of intermediate and college algebra. It is hoped that the foregoing pages will in some measure have set a pattern for the study of such problems. It is hoped further that the discussion will encourage teachers to apply themselves assiduously to the task of specifying the major relationships and concepts to be developed in connection with each topic and of discovering the main characteristic difficulties which the study of each topic presents to the students.

Among the topics which are of first-rate importance at their appro-

priate levels but which it has not been possible to discuss in this chapter may be mentioned the binomial theorem; special properties of functions and equations of degrees higher than the second; inequalities; permutations and combinations; probability; determinants; mathematical induction; logarithmic and exponential functions; general theorems on algebraic functions and equations; and the mathematics of investment and insurance. In concluding this chapter, a few general suggestions may be made with a view to helping teachers plan their instruction in these and other topics effectively.

Perhaps the most important objective of mathematical instruction at the higher levels is the development of the ability to understand *generalized* principles and concepts and to apply these generalizations properly to particular situations or problems. At the same time this seems to present to students greater difficulty than almost anything else. It is therefore a matter to which special attention needs to be given very consistently. The following examples may be cited by way of illustration.

1. The recognition of type forms in factorable expressions which in themselves may be rather complicated, such as

$12y^3 - 4y^6 + x^{2a} - 9$	(difference of two squares)
$a^3 + 8b^3 + c^3 + 6a^2b + 12ab^2$	(sum of two cubes)

2. The recognition of possibilities for reducing equations or expressions to standard forms which can be more readily handled, such as

$x^{16} + 5x^8 + 6 = 0$	(quadratic equation in x^8)
$x^4 + x^2 + 1 = 0$	(factorable by difference of two squares)

3. Determination of the nature of the roots of a quadratic equation from investigation of the character of the discriminant

4. Understanding the generality of the relations given by the remainder theorem, the factor theorem, the fundamental theorem of algebra, the theorems giving relations between roots and coefficients, etc.

5. Understanding the significance of the procedures used in solving and checking literal equations and formulas

6. Applying the formula for the rth term in a binomial expansion for a given value of r, and similarly applying other generalized formulas to the determination of values in particularized cases

It is important that instruction be carried on at suitable levels of difficulty and expectation. Nearly all textbooks in eleventh- and twelfth-grade algebra include topics and treatments which are appropriate for college algebra but are beyond the ability of the even-better-than-average eleventh-grade student. It is unreasonable to expect from the average eleventh-grade student the degree of mastery of these

topics which may legitimately be expected of superior twelfth-grade students or those taking college algebra. There is an unwarranted, though not unnatural, tendency among teachers to assume that all the material in a textbook is suitable for the students in the grade in which the textbook is used. In connection with any given topic the teacher should try to decide how far he can profitably carry his particular class in relation to that topic and what should be the nature and difficulty of the exercises which he should use and of the assignments he should make. It is to be expected that a cyclic treatment of topics in algebra will generally produce better results than too intensive treatment, carried beyond appropriate levels, in any one year. Thus, instead of trying to teach all of factoring in the ninth grade, a few simple and easily understandable cases are given in that grade. In the eleventh grade these are reviewed and more difficult ones considered, and this process is repeated in subsequent years. The work should always challenge the best efforts of the students, but, if carried beyond their capacities, it will lose its meaning and value.

Finally, the elements of time, practice, review, application, and maintenance are of extreme importance. They are all involved in a program of instruction that looks to a fundamental mastery of algebra. The great generalities do not emerge in an instant or a year. They are the result of a long process of assimilation and familiarization. They require not hours but years of concentrated attention and sheer hard intellectual work, beginning with simple concepts and proceeding to ever more difficult and abstract relationships. Rome was not made in a day, nor is there any quick, easy short cut to a real mastery of algebra. Persistent review and practice, both in the skills of algebra and in their application, is required. Otherwise they will deteriorate through disuse.

Under a program of subject matter selected to present a reasonable challenge to ability, to provide a gradual expansion of the horizon of mathematical understanding, and to demand an incessant program of maintenance, the student may expect to arrive at a *real mastery* of algebra. Such a mastery of appropriate subject matter of algebra not only will provide that student with a sound basis for exploring mathematics at its higher levels, but also will give him a richness of insight into the related fields of science.

Exercises

1. Point out any similarities and any differences between the legitimate objectives of a second course in algebra and the aims of first-year algebra. Justify the differences if there are any.

2. Consult several textbooks for the second course in algebra, and set down what the authors appear to regard as the main objectives for this course. How consistent are the different authors in their statements or implications of these major objectives?

3. Repeat exercise 2 with reference to college algebra.

4. What topics would you include in the first semester's work of a two-semester program for a second course in algebra?

5. What additional topics would you include for the second semester's work?

6. The usual way of beginning the second course in algebra is to give a concentrated and extensive review of the fundamental operations. What are the reasons for this practice? What disadvantages does it entail? Can you suggest a better plan?

7. Make a list of the specific outcomes at which you would aim in teaching quadratic equations in this second course in algebra.

8. What specific difficulties would you expect students to encounter in their work with quadratic equations? In what specific ways could you help them avoid or overcome these difficulties?

9. Construct a diagnostic test designed to locate and identify such of these specific difficulties as may persist even after the unit has been studied.

10. Do you think logarithms should be taught in the course in algebra or in connection with the trigonometry course? Why?

11. What particular things would you stress in teaching logarithms and exponentials?

12. The invention of logarithms forms an extremely interesting chapter in the history of mathematics. Prepare and give a fairly full account of this mathematical development.

13. What particular difficulties would you expect students to have in connection with the study of imaginary numbers?

14. Make a list of the specific things at which you would aim in work with graphs in a second course in algebra. If these differ from your objectives in ninth-grade algebra, explain and justify the differences.

15. What particular outcomes should be sought in work with radicals? What specific difficulties can students be expected to have in connection with this work?

16. Why is it that many students have difficulty in writing out correct values for $f(12)$ and $g(h + 3)$ when $f(x)$ and $g(x)$ are defined by such expressions as $f(x) = x^3 - 5x + 28$ and $g(x) = 12 - 5x + x^2$?

17. What steps would you take to remove such difficulties?

18. Make a well-graduated list of 30 practice exercises designed to correct the mathematical deficiency described in exercise 16.

19. Explain the difference between Horner's method and Newton's method for obtaining approximations to irrational roots of equations. Which would you prefer to have your students use? Why?

20. Explain what is meant by mathematical induction, and give an illustration of it. List the essential steps involved, and tell precisely what difficulties you would expect students to have in connection with this topic.

21. Prove the remainder theorem, and show that the factor theorem is but a special case of the remainder theorem. Do you think students generally get the full significance of the remainder theorem? If not, why don't they?

22. Compare the treatments of ratio, proportion, and variation in several textbooks in algebra. Which do you like best? Why? Be explicit.

23. Enumerate and illustrate the main properties of inequalities. Take three high-school textbooks, and see how many of the three give any treatment of inequalities.

24. The study of inequalities often receives no attention in high-school algebra and very little attention in college algebra. What do you think is the reason for this? Criticize or justify this practice, and make recommendations respecting this topic.

25. Compare the treatment of progressions in three different textbooks in college algebra. Which do you like best? Why? What do you think should be included in the study of this topic in college algebra?

26. By reference to a textbook on the mathematics of finance, show that progressions and exponential equations have important applications in that subject.

27. Make a list of the specific difficulties which students may be expected to encounter in connection with the study of one of the following topics, and be prepared to make suggestions for avoiding or overcoming these difficulties:

Progressions	Factoring
Logarithms	Quadratic functions and equations
Irrationals	Systems involving quadratics
Determinants	Binomial theorem
Inequalities	Permutations and combinations
Probability	Partial fractions

28. See if you can find, in textbooks on algebra, 10 illustrative cases in which you think students might have trouble in getting clear interpretations because of ambiguities or other reading difficulties

29. Make up three assignments of special topics or problems that would be suitable to give as honor work to very superior students in high-school algebra

30. Do the same for college algebra.

31. Write up a critical review of one textbook designed for the second course in high-school algebra.

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CHAPTER XV

GEOMETRY IN THE SECONDARY SCHOOL

As a school subject geometry has had a long and enviable career. Many virtues have been ascribed to it, some of them illusory, no doubt, and some of them magnified beyond the confines of actuality, but all bearing evidence to the respect in which the subject has been held.

Prior to and during the first quarter of the century geometry was taught in high school mainly for its alleged disciplinary value; nobody thought much about questioning the validity of this objective. Investigations made during the first two decades of this century, however, brought out conflicting and disturbing evidence with respect to the disciplinary values of all traditional educational subjects. Narrow conceptions of the "practical values" of education and certain interpretations of "transfer of training" have combined with poor teaching to produce skepticism, particularly among certain groups, as to the continuing validity of the prestige which geometry should hold as a school subject. This skepticism has been accentuated by other important concomitant factors, among which may be mentioned the unprecedented increase in high-school enrollment with the attendant lowering of the average intellectual fiber, the changing concept of the function of secondary education, the advent of the junior high school, the influence of the "activity school," and the demand for functional curricula. These phenomena have been clearly reflected in the objectives and content of all mathematics courses. The demand for a more functional type of education has led to the introduction of many new subjects into the curriculum and to an increased emphasis upon electives. It is undoubtedly true today that in a majority of high schools, or at least for a majority of high-school students, geometry is no longer a required course. Indeed, not a few educators would possibly advocate its complete exclusion from the senior-high-school curriculum.

The principal criticisms that have been directed toward instruction in geometry may be summarized as follows: (1) failure to establish a concrete basis for demonstrative geometry; (2) lack of opportunity for immediate application of the principles learned; (3) failure to organize subject matter from a psychological point of view; (4) failure to

emphasize training in logical thinking; (5) failure to correlate geometry with other school subjects; and (6) failure to recognize individual differences. An examination of texts and courses of study reveals the fact that definite efforts are being made to meet these criticisms. The teacher of geometry should be familiar with current trends in the selection and organization of geometrical subject matter. In the light of these trends and of a basic philosophy of sound mathematical instruction, he should be able to formulate aims and objectives for the teaching of geometry that would define more clearly the place of geometry in the secondary-school curriculum and delineate more explicitly its function as an instructional medium.

The Function of Geometry in the Junior High School. Against the background of historical perspective geometry occupies a place alongside arithmetic in its contribution to the basic needs of mankind. From time immemorial man has found it necessary to measure as well as to count. He has found that the concepts of size, shape, and position are ever prominent in the pattern of his environment and that "the geometric principles of equality, symmetry, congruence, and similarity are implanted in the very nature of things."[1] Primitive people obtained their first knowledge of geometry from natural objects and practical arts. The needs that developed out of art, architecture, surveying, and measurement proved to be the principal stimuli to the development of geometry before the time of Euclid. Through its excellence as a vehicle of postulational thinking, Euclidean geometry lost a great deal of this early historical significance. In recent years, however, teachers of mathematics and curriculum workers have begun to realize that "nature and the practical arts are the primary and permanent sources of geometric learning."[2]

The geometrical experience of the pupil entering the junior high school has been somewhat limited and quite casual. This experience has extended from the more or less incidental perceptions of size, shape, and position that take place during preschool days to the more systematic treatment of measurement, drawing to scale, and calculation of perimeters, areas, and volumes in the arithmetic of the sixth grade. The geometrical information of the preschool and elementary-school period has been acquired largely through manipulation and computation. It is the function of the junior high school to systematize this

[1] William Betz, The Teaching of Intuitive Geometry, *Eighth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1933), p. 59.

[2] *Ibid*, p. 100.

information and extend it to some of the broader and more general aspects of the geometry of everyday life; to aid the pupil in becoming familiar with the basic geometrical concepts and understanding the fundamental techniques, such as the use of the straightedge, protractor, compasses, and the techniques of measurement and construction; to acquaint the pupil with the characteristics of good geometrical notation; to bridge the gap from the largely manipulative type of geometric experiences to the more formal logical processes of demonstrative geometry. Such geometry has been called "intuitive," but it is rather a geometry *sui generis* which is characterized by intuition, experiment, and an informal approach to the more formal processes of demonstrative geometry. To omit any one of these three aspects would give an imperfect description of the province and function of the geometry of the junior high school.

Table 5 presents an analysis of the content of junior-high-school geometry in the perspective of desirable geometric objectives and pertinent instructional techniques. At the end of such a program of instruction the pupil should be familiar with the elementary ideas of shape, size, and position of basic geometric forms in both plane and space. He should have an appreciation of the significance of symmetry as fundamental to design and construction; of the implications of congruence and similarity in the relationships existing between geometric configurations; of the nature and importance of direct and indirect measurement; of the pertinence of dependence to the better understanding of the intrinsic characteristics of various fundamental geometric figures; and of the need for a more substantial verification of facts and establishment of truths than mere measurement, observation, or experimentation.

The Function of Geometry in the Senior High School. The informal geometry of the junior high school will constitute the complete geometrical program of many pupil . For those who continue their study of geometry in the senior high school, intuition and experiment will still be an effective aid, but the major purposes of instruction will be to instill in the pupils an appreciation for the significance of logical demonstration; to acquaint them with effective methods of clear, impartial thinking, critical evaluation, and intelligent generalization; to train them in the techniques of discovery of truth; and to introduce them to the meaning of mathematical rigor and precision.

Geometry achieves its highest possibilities if, in addition to its direct and practical usefulness, it can establish a pattern of reasoning; if it can develop the power to think clearly in geometric situations, and to use the same dis-

TABLE 5. CONTENT OF JUNIOR-HIGH-SCHOOL GEOMETRY

Vocabulary	Instructional purposes	Instructional methods
Geometric Objective—Line Segment		
broken line curved line distance equal segments graph horizontal intersecting length line line segment mid-point oblique parallel perimeter perpendicular perpendicular bisector ratio scale straight line transversal "units of length" vertical	1. To acquaint the pupil with meaning and use of lines and line segments	1. Observing lines in classroom, on models, in pictures, in geometric design, on athletic field. Introduce ideas such as "as the crow flies," the shortest distance between two points
	2. To teach the techniques of drawing and measuring line segments a. To develop reasonable competence in estimating lengths	2. Illustrate and emphasize the use of straightedge in drawing a straight line; the use of the ruler, compasses, and squared paper in measuring and comparing lengths; the use of both English and metric units of length; the approximation of lengths and distances; the approximate nature of measurement; the use of drawing instruments in making simple geometric diagrams
	3. To teach drawing to scale	3. Draw to scale simple diagrams, such as the classroom, gymnasium, football field, baseball field, basketball and tennis courts. Draw many graphs in which the student must select an appropriate scale
	4. To instill an appreciation of the need for careful and significant notation	4. Show the advantage of both types of notation, *viz.*, lettering both end points and using a single small letter to represent the line segment. Emphasize the significance of simple notation in algebraic discussions relating to line segments
Geometric Objective—Angle		
acute angle adjacent angles alternate interior angles angle	1. To teach the concept of what an angle is; both the static and dynamic aspects of angle	1. Use of board compasses, shears, hands of clock, or some similar instrument to develop the concept of an angle as the amount of opening between two lines

TABLE 5. CONTENT OF JUNIOR-HIGH-SCHOOL GEOMETRY.—(*Continued*)

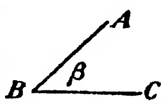
Vocabulary	Instructional purposes	Instructional methods
base angles of a triangle complementary angles corresponding angles degree equal angles exterior angle inscribed angle obtuse angle opposite angles parallel perpendicular protractor right angle side of angle straight angle supplementary angles vertex vertical angle	2 To teach pupils to recognize and classify different types of angles	2 Observation and discussion of angles found in the environment Development of the concepts of acute angle, right angle, obtuse angle, and straight angle
	3 To teach pupils the technique of angular measurement *a* To develop reasonable competence in estimating sizes of angles *b* To teach use of protractor	3 Frequent use of the protractor in measuring and drawing angles of various sizes Comprehension of what is meant by "size of an angle" Construction of simple designs involving angles of various sizes. All right angles are equal
	4 To instill an appreciation of useful and significant notation	4 Develop three types of notation for the angle, *viz*, $\angle ABC$, $\angle B$, $\angle \beta$ Show advantages of each 
	Geometric Objective—Fundamental Constructions	
altitude angle bisect bisector circle compasses construct equal equilateral line segment median mid-point parallel perpendicular perpendicular bisector protractor right angle straightedge triangle	1 To bisect a line segment	1 Make several constructions and use compasses to compare line segments Draw designs which make use of construction Show that points on bisector are at equal distances from extremities of line segment
	2 Construct a perpendicular to a line (*a*) at a point on the line and (*b*) from a point outside the line	2 Make several constructions Measure the resulting angles to show that they are right angles Bring out that construction is really drawing perpendicular bisector of a portion of the line segment Draw designs using these constructions
	3 To bisect an angle	3 Make several constructions, and use protractor to compare

TABLE 5. CONTENT OF JUNIOR-HIGH-SCHOOL GEOMETRY.—(*Continued*)

Vocabulary	Instructional purposes	Instructional methods
		resulting angles. Obtain angle of 45 degrees from one of 90 degrees, etc. Drop perpendiculars from points on bisector to sides of angle, and use compasses to compare these lengths. Draw designs using this construction
	4. Construct an angle equal to a given angle	4. Make several constructions. Check each by the protractor and some by cutting the angles out and placing one on the other to see if they fit
	5. To make applications of the fundamental constructions	5. Construct angles of 60, 45, 30 degrees, etc. Construct equilateral triangle, regular hexagon, parallel lines, circular graphs. Construct altitudes, medians, bisectors of angles, or perpendicular bisectors of sides of triangles. Divide circle into three, four, six, or eight equal parts
	Geometrical Objective Circle	
arc area center central angle chord circle graph circle circumference compasses concentric diameter equal inscribed angle pi (π) radius secant segment of circle tangent	1. To teach the concept of circle and to develop an appreciation of the extent to which circles are used 2. To teach the definition of circle and the formulas for circumference and area	1. Observation of circles in art, architectural designs, nature, machinery, clothing designs etc. 2. A circle is a curved line, all points of which are the same distance from the center. The radii of a circle are equal. Develop an understanding of radius, diameter, circumference as the distance around the circle (circumference and perimeter should be related). To develop an understanding of the relationship between diameter and circumference have pupils measure the diameters

TABLE 5. CONTENT OF JUNIOR-HIGH-SCHOOL GEOMETRY.—(*Continued*)

Vocabulary	Instructional purposes	Instructional methods
central angle locus		and circumferences of circular objects such as cans, plates, saucers, watches, etc., and calculate the ratio to three or four significant digits in each case. The formula should then be set up. The formula for area should be merely stated without development. There should be many applications. The circle as the path of a moving point
	3 To teach the application of concepts and formulas related to circles in problems of mensuration	3 Use many varied instances of circles in design and construction
	4. To teach the technique of drawing circles	4. Teach use of compasses and string (at board). Point out that, when using a string to draw a circle on the blackboard a right-handed person should always start with right hand as far around to the left as possible and draw the circle clockwise keeping the chalk perpendicular to the string at all times (a left handed person would want to draw the circle counterclockwise starting with left hand as far around to right as possible). Draw many designs using circles
	5 To study positional relationships of line and circle, and of two circles	5. A line may contact a circle in no points, one point, or two points Discuss the significance of each position. Emphasize the fact that there are at most two points of contact Two circles may be in one of six possible positions with respect to each other; concentric, tangent (two-positions),

TABLE 5. CONTENT OF JUNIOR-HIGH-SCHOOL GEOMETRY.—(Continued)

Vocabulary	Instructional purposes	Instructional methods
		intersect, not intersect, coincident. Emphasize that there are at most two points of contact for two distinct circles
Geometrical Objective—Polygons		
acute-angle triangle altitude area base congruence congruent diagonal equiangular equilateral formula hexagon isosceles trapezoid isosceles triangle obtuse triangle octagon parallelogram perimeter polygon quadrilateral rectangle regular rhombus right triangle scalene triangle similar similarity square symmetry trapezoid triangle vertex (vertices)	1. To acquaint pupils with the fundamental properties of triangles	1. Analysis of the various types of triangles and the part angles play in their make-up. Sum of the angles equals 180 degrees. Study of congruence and similarity through experiment. Constructing and fitting triangles of the three types (a.s.a., s.a.s., s.s.s.) to test for congruence and (a.a.a.) for similarity. Relations between corresponding parts of congruent and similar figures. Ratio and proportion. Shadow ratio in right triangle. Nature and properties of isosceles, equilateral, and right triangles. The angles of an equilateral triangle are all equal; the base angles of an isosceles triangle are equal. Pythagorean relation
	2. To acquaint pupils with the fundamental properties of quadrilaterals	2. Sum of angles of quadrilateral equals 360 degrees. Nature and properties of parallelogram, rectangle, square, rhombus, and trapezoid. Similarity and congruence. Ratio and proportion. Parallelism of sides. Base and altitudes
	3. To study characteristics of certain regular polygons	3. Equilateral triangle, square, hexagon, octagon. Construct many regular polygons, and use designs which emphasize them. Emphasize equality of angles and of sides
	4. To apply the fundamental principles of	4. Study of perimeters and areas with establishment of basic

TABLE 5. CONTENT OF JUNIOR-HIGH-SCHOOL GEOMETRY.—(Continued)

<table>
<tr><th>Vocabulary</th><th>Instructional purposes</th><th>Instructional methods</th></tr>
<tr><td></td><td>measurement

5. To develop an appreciation for, and an ability to use, significant notation</td><td>formulas. Develop fundamental understanding of each concept and its calculated value

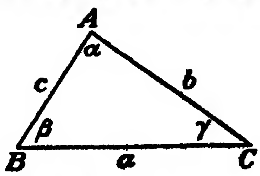
5. Call particular attention to the triangle, in which capital letters represent vertices, and the corresponding small letters the opposite sides. It is desirable to use small Greek letters for the angles. The use of capital letters for vertices of all polygons should be emphasized. The use of b for base, h for height, A for area, and p for perimeter is recommended. In

the case of trapezoids it is recommended that b_1 and b_2, or b and b', be used to represent the two bases</td></tr>
<tr><td colspan="3">Geometrical Objective—Indirect Measurement</td></tr>
<tr><td>angle
angle mirror
angle of depression
angle of elevation
congruent
distance
length
ratio
ruler
sextant
similar
tape measure
"the more commonly used units of measurement"</td><td>1. To find distances and angles by congruent triangles:
a. To become familiar with the congruency of triangles because of the equality of two angles and an included side
b. To become acquainted with the congruency of triangles because of the equality of two sides and an included angle</td><td>1. Construct and test for congruence several triangles with equality of s.a.s. and a.s.a. Determine lengths of inaccessible distances (through buildings, across swamps, etc.). Use surveying instruments, and make constructions with ruler and compasses. Further practice in measuring lengths and angles</td></tr>
</table>

TABLE 5. CONTENT OF JUNIOR-HIGH-SCHOOL GEOMETRY.—(*Continued*)

Vocabulary	Instructional purposes	Instructional methods
transit	2. To find distances and angles by means of similar triangles: a. To become familiar with the similarity of triangles because of the equality of angles b To become familiar with the similarity of triangles because of ratios of corresponding sides c. To become familiar with the concept of the "shadow ratio" as a basis for the better comprehension and use of the tangent ratio	2. Construct and test for similarity several triangles whose angles are equal, also whose sides are proportional. Find the heights of flagpoles, trees, buildings. Develop the concepts of "angle of elevation" and "angle of depression." Develop the use of the "shadow ratio" (the ratio of an object to its shadow) as a technique for using a known height to find an unknown height. Point out the constancy of the angle for equal shadow ratios
	3 To find distances and angles by scale drawings	3. Develop the use of ratios in scale drawings and emphasize the convenience of the use of graph paper in drawing to scale. Make scale drawings of many familiar designs
	Geometric Objective—Solids	
altitude cone cube cylinder diagonal edge face formula length, width, height plane prism pyramid	1. To develop the ability to draw solids and see solids in proper perspective	1. Point out some of the elementary aspects of looking at objects in perspective (looking down a railroad track, looking at pictures of rooms and their contents, the movie screen, etc.). Practice drawing the simpler solids so that they really look like what they are supposed to be. Emphasize that lines that are behind planes should be lighter than others and dotted, etc. If

TABLE 5. CONTENT OF JUNIOR-HIGH-SCHOOL GEOMETRY.—(*Continued*)

Vocabulary	Instructional purposes	Instructional methods
rectangular solid space sphere surface surface area "units of volume" vertex volume	 2. The recognition of solids and their importance 3. The study of properties of solids	colors are used all lines lying in the same plane should be of same color 2. Seek for pictures of such solids, look for them at home, at school, at work, at play. (For example: the chalk box, the classroom, church steeples, silos, baseballs, basketballs, tennis balls, etc.) 3. Build up through experiment the relation between corresponding parts, formulas for surface areas and volumes. Bring out the various plane sections
	Geometric Objective—Geometric Dependence	
change dependence formula ratio variation varies	1. To acquaint the pupil with some of the more elementary and yet fundamental aspects of dependence in the study of geometry	1. Use the various formulas for perimeter, area, and volume as background for raising pertinent questions concerning the behavior of one aspect of a geometric situation for given changes in other associated aspects of the same situation
	Geometric Objective—Informal Reasoning	
axiom conclusion definition hypothesis postulate reason	1. To bridge the gap between the intuitive and experimental geometry of the early junior high school and the more formal geometry of the senior high school	1. Give many opportunities for the pupils to replace direct measurement and observation by simple inferences. Try to develop an appreciation for the need of logical reasoning. Show the importance of analysis of situations to bring out the implications involved
	Geometrical Objective—Numerical Trigonometry*	
angle degree	1. To develop the ability to sketch good	1. Explain and discuss the meanings of the trigonometric func-

* This will be discussed more fully in Chap. XVII.

TABLE 5. CONTENT OF JUNIOR-HIGH-SCHOOL GEOMETRY.—(Continued)

Vocabulary	Instructional purposes	Instructional methods
right angle acute angle right triangle hypotenuse leg	working drawings for problems involving trigonometric solution from conditions stated verbally	tions, and show, by various illustrations, how they are used to determine distances and angles without actual direct measurement
opposite side adjacent side similar ratio proportion scale drawing	2. To develop an understanding of the meaning of the sine, cosine, and tangent of an angle	2. Strengthen the students' understanding of these meanings and applications by giving varied type problems to be solved by trigonometric functions
sine cosine tangent height distance horizontal vertical	3. To develop an interpretation of each of these functions in each of the following two ways: a. As a ratio b. As a numerical multiplier	3. Discuss and illustrate how trigonometric methods are used in practice by surveyors and engineers
	4. To develop the ability to use tables of sines, cosines, and tangents	4. So far as possible, have the students carry on appropriate field problems, using instruments to collect their own primary data and using trigonometric methods to obtain final results
	5. To develop the ability to use trigonometric functions to determine distances and angles without actual measurement	
	6. To develop an appreciation of the method and usefulness of trigonometry	

crimination in non-geometric situations; if it can develop the power to generalise with caution from specific cases, and to realize the force and all-inclusiveness of deductive statements; if it can develop an appreciation of the place and function of definitions and postulates in the proof of any conclusion, geometric or non-geometric; if it can develop an attitude of mind which tends always to analyze situations, to understand their interrelationships, to question

hasty conclusions, to express clearly, precisely, and accurately non-geometric as well as geometric ideas.[1]

From a study of demonstrative geometry every pupil should acquire a comprehensive knowledge of geometric facts, concepts, and processes; an intimate acquaintance with the nature of deductive reasoning, not only as an integral part of a coherent logical structure but also as an effective method in detached arguments and in the discovery of truth; and some facility in the application of geometry to the better interpretation and appreciation of one's environment To the attainment of these ends, analyses of textbooks indicate definite trends in the direction of more significant pedagogical arrangement of geometrical subject matter.[2] These trends may be summarized as follows:

1. Increasing emphasis is given to geometrical instruments and their use.
2. Values to be gained from a study of geometry are pointed out more frequently.
3. Formulas are derived and used more frequently
4. General and historical notes are given more frequently.
5. Symbols and abbreviations receive more general use.
6. Summaries are included more often.
7. Illustrations are used more frequently
8. In the more recent books there are to be found topics from other branches of mathematics.
9. Recent books include topics dealing with practical applications of geometric principles.
10. Simpler language and less formal style are used.
11. Varied typography, the parallel form of presenting theorems, schematic forms for summaries, and indexes are more common features in recent books.
12. There is a decrease in the number of terms defined; the number of assumptions is increased materially, there has been a decrease in the number and formality of theorems, problems, and corollaries, the analytic treatment of theorems, developmental exercises, and exercises for immediate application have become increasingly important innovations; exercises requiring rigorous geometric proof have decreased in number while computational exercises (those requiring applications of arithmetic, algebra, and numerical trigonometry) have shown an increase.
13. More informal introductions are characteristic of the newer books.
14. In recent books there is less attention given to scientific rigor and

[1] H. C. Christofferson, "Geometry Professionalized for Teachers" (Oxford, Ohio: published by author, 1933), p 28

[2] Dell Terry: "An Analysis of Some Plane Geometry Texts" (Nashville, Tenn.: unpublished M. A thesis, George Peabody College for Teachers, 1932).

E. M. Freeman, Textbook Trends in Plane Geometry, *The School Review*, 40 (1932), 282-294.

strict logic and more given to the presentation of geometric subject matter from the point of view of the immature pupil.

15. To give emphasis to the need for training in logical thinking, more use is being made of the incomplete proof.

16. More attention is given to the problem of individual differences through optional material and differentiation of materials.

The Status of Solid Geometry. The status of solid geometry in the secondary school has caused rather general dissatisfaction among teachers of mathematics for several years. It is not merely a selfish attitude which prompts this feeling, but a genuine interest in the welfare of the pupils concerned. Only in the larger systems is solid geometry still offered regularly, and there the course is elected only by a very few students. Thus there is the situation in which considerable time is spent studying two-dimensional space with very little attention being given to the three-dimensional space in which we live. It seems that solid geometry, a study of space and objects about us, has a practical aspect which should not be neglected, regardless of whether or not the student intends to study more mathematics.

Realizing that the ratio of the number of pupils studying solid geometry to the number of those studying plane geometry is still decreasing, even though that ratio is already very small, many mathematics teachers advocate that essential elements of both courses be taught in the same year. This idea has been discussed by a number of prominent writers in the field of the teaching of mathematics. Nearly all admit the possibility of combining the courses. But the desirability of doing so, and the methods to be used, have been subjects of controversy for a number of years.

From suggestions of similarities between plane and solid geometry it is evident that the most common conception of the combined course among those who see it as a "fused" course is that analogies should be prominent. Similarities which are frequently mentioned include perpendiculars to lines and planes, parallel lines and planes, congruence and similarity in plane and solid figures, plane and spherical triangles, bisectors, plane and dihedral angles, circles and spheres, parallelograms and parallelepipeds, polygons and prisms, loci, areas of plane figures and areas and volumes of solids, and the solid figures generated by moving plane figures through space.

The opinion is expressed frequently that, although a certain development of solid geometry should be carried along with the plane, no formal proofs should be required for the solid geometry theorems in such a course. Although numerous teachers of mathematics have

endorsed such a plan, it is not to be inferred that they are in agreement as to the proper extent or method of treatment.

When a one-year course was first seriously considered, the usual objection was that it would not meet the requirements of the College Entrance Examination Board. That objection was removed in 1923 when the Board published Document 108 in which the following declaration was made:

The Board wishes to accord all due latitude in the treatment of the subject of solid geometry. It recognizes the value of the further training in logical demonstration which supplements the study of plane geometry and is given in the standard courses at the present time. It recognizes also that the intuitive geometry of the early school course may well be carried further as regards both a firmer grasp on space relations and the visualization of space figures, and the mensuration of surfaces and solids in space.[1]

In this document there was also announced a new college-entrance examination to be offered. It was known as Mathematics *cd* and was intended to cover a one-year unit of plane and solid geometry. This course has not become popular. Less than 100 took the examination during the first six years it was offered, and there is some evidence that some of these took it because they did not know enough geometry to know which examination to take. Probably one reason that this course has not developed is that there has been no suitable textbook.

Also the National Committee on Mathematical Requirements stated in its 1923 report: "The aim of the work in solid geometry should be to exercise further the spatial imagination of the student and to give him both a knowledge of the fundamental spatial relationships and the power to work with them."[2]

It will be noted that no mention is made of a form of proof in the statement of the aim. There are other statements in the report which indicate a favorable attitude toward teaching material from both two- and three-dimensional geometry in the same year, *viz*,

If the student has had a satisfactory course in intuitive geometry and some work in demonstration before the tenth grade, he may find it possible to cover

[1] College Entrance Examination Board, *Document* 108—Definition of the Requirements in Geometry (New York: College Entrance Examination Board, 1923), p. 6. (Note to the reader: This quotation does not necessarily represent the present attitude of the College Entrance Examination Board The present policy of the Board is given in its document "Description of Examination Subjects," edition of December, 1940) See p 33.

[2] National Committee on Mathematical Requirements, "The Reorganization of Mathematics in Secondary Education" (Boston Houghton Mifflin Company, 1923), p. 52.

a minimum course in demonstrative geometry, giving the great basal theorems and constructions, together with exercises, in the 90 periods constituting a half year's work. . . . [1]

Some schools will find it possible and desirable to introduce the more elementary notions of solid geometry in connection with related ideas of plane geometry.[2]

In 1929 a committee was appointed jointly by the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics to study the feasibility of a proposal to alter the college-entrance requirements in geometry so as to bring about more extensive introduction of courses including the essentials of plane and solid geometry in a single year's work in place of the traditional year of plane geometry.[3] This committee made three reports. The first two were concerned chiefly with the membership of the committee, a statement of the problem, and the plan of attack. The third report, which was quite detailed, will be discussed later.

The 1930 meeting of the National Council of Teachers of Mathematics was devoted almost entirely to the teaching of geometry. One of the principal topics of discussion was the idea of a one-year course in plane and solid geometry. In the same year the Council published the *Fifth Yearbook*, entitled "The Teaching of Geometry." Of 14 articles in the book two were given entirely to the problem of a combined course, while significant sections of four others dealt with the same topic.

A second committee was appointed by the Council to study the details in the proposed change with the idea that teachers should have some definite recommendations. Definiteness was somewhat lacking in the report of this committee as there was no commitment for or against the program. Opinions of people who had had some experience in combining the courses were collected and summarized and two tentative courses for experimentation were suggested.

The Committee gave a rather inclusive summary of the reasons for and against a combined course, the ideas of which are given below.

Reasons for a Combined Course

1. It is unfortunate that only 10 per cent of the pupils taking plane geometry study the geometry of the three-dimensional world in which we live.

[1] *Ibid.*, p. 50.

[2] *Ibid.*, p. 52.

[3] D. Jackson, *et al.*, Report of the Committee on Geometry, *The Mathematics Teacher*, 24(1931), 298–302.

2. The fused course is more interesting because of more genuine illustrations.

3. All geometry is a single subject.

4. Simpler parts of both kinds of geometry could be put in a tenth-year course and the more complex parts put into a later elective course.

5. Definitions and concepts would be broadened.

6. Formulas of solid geometry should form a part of our general knowledge.

7. The third semester course might include some elementary notions of the calculus.

REASONS AGAINST A COMBINED COURSE

1. Tenth-year pupils are not mature enough to study solid geometry.

2. The experimental basis is not strong enough yet.

3. Sufficient intuitive geometry is not given in our elementary schools for the foundation of such a course.

4. Our present plane geometry course would be wrecked.

5. Many of our high school pupils have difficulty even with the present course in plane geometry.

6 A superficial course in the tenth year would take the zest out of the course.

7 A course including both types of geometry would interfere with the cultivation of postulational thinking

8. In order to complete the courses in one year too many assumptions are necessary. It is wrong to assume difficult theorems in order to prove a few easy exercises

9. New and undefined courses work a hardship upon pupils who transfer from one school to another.[1]

The Committee on Geometry, of which Prof. Ralph Beatley was chairman, published its third report in 1935 [2] This is the most comprehensive piece of work which has been done on the status of the proposed combined course There was no very definite position taken by the committee as a whole on the question of the one-year course. The report was quite general on the teaching of geometry instead of adhering strictly to its purpose as stated in the first report.

Ideas were gleaned from committee reports, magazine articles, books on method, and textbooks The members of the committee expressed their opinions on these ideas, and the opinions of 101 teachers in service were collected.

Except for a very small minority there was general agreement that more concepts built up in intuitive geometry may be postulated in the

[1] Charles M Austin, *et al*, Report of the Second Committee on Geometry, *The Mathematics Teacher*, **24** (1931), 370–394

[2] Ralph Beatley, Third Report of the Committee on Geometry, *The Mathematics Teacher*, **28** (1935), 329–379, 401–450.

demonstrative course if the pupils understand the nature of postulates. They must understand that, if something different were assumed, the resulting chain of logic would be different and that, if the assumptions are false, then conclusions based upon them are also false. It was the consensus that the important facts of geometry can be mastered in the junior high school through inductions in intuitive geometry. Opinion was almost equally divided on the suggestion that informal solid geometry should include ideas concerning parallel planes cut by transversals, plane sections of solids, and perpendicular lines and planes. It was felt that nothing should be done to sacrifice the emphasis on logical thinking and the use of geometry as a vehicle for training in the technique of logical thinking.

Regardless of the method of selecting and organizing the content of the revised course, it would probably not be logically complete. The assumption that there is a certain amount of material which must be covered is false. We do not now *finish* geometry—one could not do that in a lifetime. The fact that the course is logically incomplete need not cause any distress if everyone is aware of the fact. The important thing is that the pupils understand the meaning of a logically complete system and that they know the difference between a good and a poor proof. It is not a matter of the present course absorbing solid geometry—it is a question of whether or not omissions can be made to care for the new demands without sacrificing the essential values of the present course. Besides eliminating some of the present material, the change would require simultaneous introduction of the fundamentals of both plane and solid geometry.

Four methods which have been recommended for effecting this simultaneous treatment of plane and solid geometry are as follows: (1) introduction to abstract geometry through concrete experiences in the real world of three dimensions [the hypothesis underlying this suggestion is that a more generalized (three-dimensional) treatment will be more effective than a restricted (two-dimensional) treatment for developing general ideas of such fundamental concepts as congruence, symmetry, similarity, locus, geometric forms, angles, parallelism, perpendicularity, motion by rotation and translation, position referred to the x-y-z-axes, size, and measurement of length, area, volume, and angles]; (2) the extensive use of analogy; (3) the consideration of related planes in different positions, as in the surfaces of various solids; (4) the allocation of a considerable amount of time near the end of the course to the study of mensuration, the criterion for the selection of the prob-

lem material being the significance of the material, without regard to dimensionality, to normal fifteen-year-old children.[1]

Efforts to outline such courses in syllabuses fall into three general classes: (1) "tandem" courses, (2) analogy and intuition, and (3) a course in which similar theorems from both plane and solid geometry are formally developed, but the theorems for the plane are given first for each chapter or "book," and corresponding theorems for the solid follow.

The National Committee on Mathematical Requirements lists fundamental and subsidiary theorems for both plane and solid geometry (considered independently). The College Entrance Examination Board includes 96 theorems and 13 constructions, of which 33 are starred, in the Mathematics *cd* course; but the order for teaching is not indicated. In the recent specifications for examinations in mathematics, the College Entrance Examination Board does not include any solid geometry in the *alpha* examination. It states that the *beta* examination "may include questions from solid geometry" and that the main body of the *gamma* examination "will be concerned with trigonometry, solid geometry, and advanced algebra." Nothing is said concerning the fusion of plane and solid geometry.[2] Evans proposes 53 plane geometry theorems and 51 solid geometry theorems, but he does not combine them.[3] Christofferson lists what he considers the minimum essentials of both plane and solid geometry, but the lists for the plane and solid are separate.[4] The Second Committee on Geometry included two syllabuses for a combined course in their report. One is a "tandem" course and the other has the solid geometry interspersed throughout the course as imagination exercises and discussion.[5] Gertrude Allen gives a complete syllabus for the course referred to above.[6] The solid geometry is not presented as theorems. The most significant change is that locus is taught by

[1] Gertrude E. Allen, An Experiment in Redistribution of Material for High School Geometry, *Fifth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1930), p. 73–75.

[2] College Entrance Examination Board, "Description of Examination Subjects" (New York: College Entrance Examination Board, December, 1940), pp. 35–46.

[3] G. W. Evans, Proposed Syllabus in Plane and Solid Geometry, *The Mathematics Teacher*, **23** (1930), 87–94.

[4] H. C. Christofferson, *op. cit.*, pp. 10–13.

[5] Austin, *et al.*, *op. cit.*, pp. 370–394.

[6] Allen, *op. cit.*

beginning with space and then limiting it to a plane. Beatley[1] and Reeve[2] both give syllabuses in which theorems for both plane and solid are offered for particular topics, but for each topic the theorems for the plane and then for the solid are given without interweaving the two.

The Function of Geometry in the Junior College. The concept of motion and the rectangular framework of coordinates which contribute so fundamentally to the analysis of algebraic formulas and geometric configurations are the key concepts of the analytic geometry of the junior college. The primary function of geometrical instruction during this period of secondary mathematics is twofold in nature: (1) to coordinate the fundamental concepts, principles, and techniques of algebra, geometry, and trigonometry and (2) to build up a proficiency in the algebraic analysis of geometric configurations and in the geometric interpretation of algebraic expressions.

Klein's definition of geometry (*cf.* page 449) must begin to occupy the center of the stage. The pupil must be taught to analyze a geometric configuration for its characteristic invariants. He must learn to associate algebraic expressions with their proper geometric configurations and to know those invariant relations which help to characterize the configuration and distinguish it from all other configurations. The mere plotting of points and the consequent drawing of a smooth curve must cease to be the goal of attainment. Why does $y = ax^2 + bx + c$ give a parabola, $b^2x^2 + a^2y^2 = a^2b^2$ an ellipse, and $b^2x^2 - a^2y^2 = a^2b^2$ a hyperbola? Why does the change of sign in the equations of the ellipse and hyperbola have so much effect upon the curves the equations represent? Why is a circle a special case of the ellipse? What does symmetry with respect to an axis, or with respect to a point, mean, and what are the characteristics of symmetry in the equation of a curve? How does one learn to recognize these characteristics, and of what value are they in interpreting the equation? These are some of the elementary, yet fundamental, questions which should be constantly stressed in any course in analytic geometry.

The general nature of the concept of dimensionality should be related to the concept of coordinates. Why are two coordinates needed to locate a point in a plane? What does it mean for a point to move with one degree of freedom? What is meant by a parameter,

[1] Ralph Beatley, Notes on the First Year of Demonstrative Geometry, *The Mathematics Teacher*, **24** (1931), 213–222.

[2] W. D. Reeve, Tenth Year Mathematics Outline, *The Mathematics Teacher*, **23** (1930), 343–357.

and what is the relationship between the concept of parameter and the concept of dimensionality? The general importance of transformations and their significance are also to be emphasized. How and when may the equations of geometric configurations be transformed for the purpose of studying them in detail? Such questions as these help to point out some of the characteristics of generalization. Students should have every opportunity to study such techniques under critical yet helpful supervision. Why can the general equation of the second degree be transformed to one of the typical forms before it is analyzed to determine what curve it is? When can special coordinates be used in setting up a general figure? How can special coordinates be used for the purpose of simplifying the analysis of a general figure? For example, in the problem: *Show that the medians of a triangle intersect in a point,* why is no generality lost in using $(0, 0)$, $(x_1, 0)$, and (x_2, y_2) as the coordinates of the vertices of the triangle instead of (x_1, y_1), (x_2, y_2), (x_3, y_3)? How does the use of these coordinates simplify the problem? Under what circumstances can such simplifications be introduced into a problem? In the above problem would the use of the more general coordinates make any specific contribution that cannot be obtained from the special coordinates?

Emphasis should be placed upon the nature and use of polar coordinates. When should they be used? What characteristics of a configuration suggest the use of such coordinates? What are their relations to rectangular coordinates, and what is the full significance of these relations? What are the characteristics of algebraic expressions that suggest the introduction of polar coordinates to simplify the discussion?

The junior-college student of geometry is a fairly mature student and is ready for more profound interpretations and more ingenious generalizations. Instruction in analytic geometry, through its effective correlation of algebra, geometry, and trigonometry, should offer the pupil the opportunity for a more significant understanding of geometry in its true mathematical perspective. It should also give him a rather thorough introduction to the elementary algebra of invariants which is one of the most efficient techniques for making careful generalizations. Finally, it should lay the foundation for an intelligent approach to the abstraction of mathematical processes introduced by the calculus.

Exercises

1. What are some of the most outstanding changes implied by current trends in the selection and organization of geometrical subject matter?

2. Select a text in high-school geometry with copyright date 1945 or later,

and list the most significant steps taken by the author to meet the criticisms summarized on pages 397 to 398.

3. Give a specific statement of your interpretation of the function of geometry in the junior high school.

4. Give a specific statement of your interpretation of the function of geometry in the senior high school.

5. In what sense should the geometry of the junior high school be experimental?

6. In what sense should the geometry of the junior high school be intuitive?

7. What is the function of direct and indirect measurement in the geometry of the junior high school?

8. Select a text in high-school geometry with copyright date 1945 or later, and list illustrations of the 16 points given on pages 409 to 410.

9. To what extent do you think the geometry of the junior high school should be used as background for the geometry of the senior high school?

10. What do you consider as the most important arguments for the fusion of plane and solid geometry?

11. What do you consider as the most important arguments against the fusion of plane and solid geometry?

12. What are the arguments for and against some work in solid geometry in the senior high school?

13. Give a specific statement of your interpretation of the function of geometry in the junior college.

14. Answer all questions raised in the section on The Function of Geometry in the Junior College (pages 416 to 417).

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CHAPTER XVI

SOME SPECIAL ASPECTS OF DEMONSTRATIVE GEOMETRY

The demonstrative geometry of the senior high school should be taught chiefly as a course in reasoning. It should aim to develop the habits of independent and careful thinking rather than strive to present the subject matter of geometry as a *finished* model of deductive thinking. It is extremely important that each pupil do his own thinking, observing, and comparing, and that new ideas, statements, truths, and theorems be discovered by the pupil himself. Teachers should therefore give thoughtful and conscientious consideration to the most effective means of attaining these goals of instruction. Each teacher should have a clear comprehension of the nature of proof and the significance of demonstration. He should know something of the relative effectiveness and the distinguishing characteristics of the various techniques of careful thinking. He should be skilled in the methods of application and generalization of geometrical subject matter.

The Nature of Proof. From the Third Report of the Committee on Geometry we have the statement that

> . . . it is generally agreed that the important facts of geometry can be mastered below the tenth grade through inductions based on observation, measurement, constructions with drawing instruments, cutting and pasting, and also through simple deductions from the foregoing inductions as well as from geometric notions intuitively held.[1]

Thus it would seem that the responsibility of senior-high-school geometry is entirely defined within the province of postulational thinking. This does not imply the mastery of a significant body of theorems any more than it implies intelligent comprehension of the methods involved in arriving at and establishing the truths embodied within such theorems. There is a definite propaedeutic value in the mastery of a number of geometric theorems which is neither to be overlooked nor minimized. In the main, however, such academic

[1] Ralph Beatley, The Third Report of the Committee on Geometry, *The Mathematics Teacher*, 28 (1935), 334.

value is overshadowed by the importance to the individual of the technique of discovery, evaluation, and establishment of truth. The clear statement of definitions, the critical analysis of assumptions, the careful weighing of evidence, and the impartial deduction of implied conclusions are invaluable contributions which the intelligent study of demonstrative geometry may make to the general education of senior-high-school pupils.[1]

One of the major problems that confront the teacher of demonstrative geometry is to teach the pupil to reason without reference to unestablished circumstantial evidence. For example, in proving a theorem related to triangles, a pupil is inclined to think in terms of a right triangle if one of the angles looks like a right angle, or in terms of an equilateral triangle if the diagram looks as if the three sides are equal. The pupil's attention should be called to the dangerous pitfalls of inaccurate reasoning that lie behind such diagrammatic camouflages. An inexperienced pupil is likely to use such special diagrams and is just as likely to draw general conclusions from specific situations. Emphasis should be given to the necessity of drawing accurate diagrams with ruler and compasses in situations that demand them, such as construction problems, prescribed written work, or any situation which places a great deal of emphasis on accuracy of figure. Freehand drawings that are approximately accurate may be used occasionally, particularly for the drawing of figures that are purely demonstration figures, such as in proofs of theorems or solutions of original exercises where the emphasis is on the method of attack. In every case the argument for the establishment of a truth should place the figure in the background, *i.e.*, only known and established facts related to the figure should be used.

The above remarks do not intend to imply that there will be no place for intuition in senior-high-school geometry. A great many of the "intuitive truths" from junior-high-school geometry should be accepted as a background for senior-high-school geometry. Furthermore, there should be opportunities for experimentation and intuitive thinking in senior-high-school geometry. The teacher must make sure that a clearly defined distinction is drawn between the techniques and functions of experimentation, intuition, and demonstration. The subject matter of geometry offers a body of materials concerning which individuals can think impartially and critically and about which they

[1] A very interesting study to read in this connection is H. P. Fawcett, The Nature of Proof, *Thirteenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1938).

can form conclusions entirely uninfluenced by their emotions. It is for this reason that the study of geometry can be used as a means for analyzing the techniques of careful argumentation and clear thinking, *viz.*, induction and deduction, synthesis and analysis, direct and indirect proof.

Induction and Deduction. The technique of making the "transition from particular facts to a general knowledge about these facts is known as 'the process of induction.'"[1] When the pupil measures the angles of several triangles and finds that in each case the sum of the angles closely approximates 180 degrees, or when he cuts out these angles and fits them together and finds that they make a straight angle, he has the background for the induction that "the sum of the interior angles of any triangle is 180 degrees." If an individual had two containers, one conical in shape and the other cylindrical but of the same height and diameter, it would be a very simple experiment to show that the conical container held only one-third as much as the one which was cylindrical in shape. The induction might possibly follow for this individual that such a relationship between the contents of a cone and a cylinder of the same dimensions would always hold. Such inductions are important in the discovery of truths whether geometrical in nature or not.

There are dangers involved, however, which should be recognized by the teacher. It is dangerous to make generalizations from specific cases alone. It is a well-known fact that the formula $p = n^2 - n + 41$ will produce a value for p which is a prime number for all integral values of n from $n = 0$ to and including $n = 40$. But when $n = 41$. $p = (41)^2 - 41 + 41 = (41)^2$, which is not a prime number. Thus here is a rule which works for 41 values of n, but the mere fact that it fails when $n = 41$ prevents the generalization. A rule may work in all cases but one, but, if one case can be found in which it fails to work, it cannot be stated as a general rule. Discoveries by induction are in the realm of probable truths. Thus by induction the pupil, who experiments with the angles of a triangle or the relationship between a cone and cylinder of like dimensions, can only say, in the respective cases: (1) "It is probably true that the sum of the interior angles of any triangle is 180 degrees" or (2) "The volume of a cone is probably equal to one-third the volume of a cylinder of the same dimensions."

Such experiments as these lay the foundations for inductive generali-

[1] Columbia Associates of Philosophy, "An Introduction to Reflective Thinking" (Boston: Houghton Mifflin Company, 1923), p. 74.

Deduction is the "process of following the network of relations which bind truths together."[1] *Deductive reasoning is thus the process of drawing logical inferences from established facts or fundamental assumptions. Demonstrative geometry is primarily a deductive* science in which truths, stated in the form of theorems, can be proved by showing that they are implied by other theorems which have already been proved, definitions that have been stated, and postulates that have been accepted. The definitions of fundamental terms should be phrased to conform with experience, experiment, and common and universal usage. The principal functions of definitions are clarity, simplicity, and brevity of expression. In the last analysis there necessarily will be certain undefined terms which are accepted as established elements of common knowledge. For example there is no clarification of concepts in setting up formal definitions of point, line, plane, and space in approaching the study of geometry. Similarly, in any deductive science there necessarily will be a basic list of fundamental assumptions. In geometry there are certain basic axioms and postulates which we accept as true and agree to use as an aid in drawing any conclusions they may imply. It should be emphasized clearly and constantly that these assumptions are merely statements accepted as true because of their conformity with common experience and sound judgment and that they are in no sense "self-evident" truths. From a pedagogical point of view the principal characteristics of such a body of assumptions are as follows:

1. Consistency—there should be no contradictory statements in the list.
2. Simplicity of statement—they should be free from ambiguous statements and should be in a form that would permit ready deductions.
3. They should present no conflict with established knowledge or observable facts.

It is evident to the thoughtful student of geometric techniques that the intelligent study of geometry is both inductive and deductive in nature. The teacher should strive to keep before the student the challenge of inductive discovery and the assurance of deductive proof.

Synthesis and Analysis. Benjamin Peirce defined mathematics as "the science which draws necessary conclusions."[2] In the statement of this definition we have the essence of synthetic argumenta-

[1] *Ibid.*, p. 98.

[2] Benjamin Peirce, "Linear Associative Algebra," Sec. 1, lithographed, 1870. Reprinted in the *American Journal of Mathematics*, 4 (1881), 97.

tion. Every geometric theorem has two characteristic properties, a hypothesis and a conclusion. The hypothesis is a statement of the accepted relationships of a given configuration which are to be used in the search for new relationships which are summed up in the conclusion of the theorem. The proof of the theorem consists in the establishment of the truth of the conclusion through implications and inferences that find their original source of justification in the hypothesis. The synthetic proof consists of the drawing of a series of necessary conclusions until the desired conclusion is reached. The hypothesis implies, as a necessary consequence, the hypothesis of some axiom, postulate, or previously established theorem; these hypotheses imply the conclusions associated with them, which in turn make further implications, and this chain of necessary deductions is pursued until the desired conclusion is reached. Although the simplicity, elegance, and rigor of this form of argument make it highly desirable, nevertheless, it is far from desirable as a sole procedure to be followed in deriving geometric proofs. As a technique it makes no provision for the pupil to understand the reason for making significant constructions or for applying different theorems. As a simple illustration let us consider a synthetic proof for the theorem:

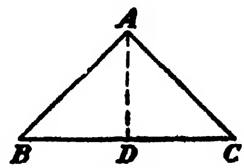

FIG 25.

If two sides of a triangle are equal, the angles opposite these sides are equal.
Given $\triangle ABC$, in which $AC = AB$.
To prove: $\angle B = \angle C$.
Proof:

1 Draw AD bisecting $\angle BAC$ and meeting BC in D	1 To bisect a given angle
2 In $\triangle$ BAD and CAD, $AB = AC$.	2. By hypothesis
3 $\angle BAD = \angle CAD$.	3 By construction.
4 $AD = AD$.	4. Common side
5 $\therefore \triangle BAD \cong CAD$.	5. Two triangles are congruent if two sides and the included angle of the one are equal, respectively. to two sides and the included angle of the other
6 $\therefore \angle B = \angle C$.	6 Corresponding angles of congruent $\triangle$.

The hypothesis of the given theorem, *through the auxiliary construction*, implies the conditions of the hypothesis of the congruence theorem (side, angle, side), which in turn implies through its conclusion the desired conclusion of the theorem, *viz*, the angles opposite the equal

sides are equal. It should be pointed out that to the inexperienced pupil the auxiliary construction of step 1, which is absolutely necessary in this particular synthetic proof, is likely to be a bolt from a cloudy geometric atmosphere followed by a crash of misunderstandings and misgivings. Why draw AD? How did the author know such a line should be drawn? Many questions such as these are likely to arise, and they are certain indicators of a lack of understanding and self-confidence in the pursuit of geometric information.

The analytic approach to a geometric proof consists in the search for sufficient conditions. The investigator looks at the conclusion of the theorem and raises the question: What relation is sufficient to justify the use of this conclusion as a true statement? Once this relation is found, he analyzes it for the same purpose, with the hope in mind that ultimately he will arrive at the hypothesis of the theorem as the source of the chain of sufficient reasons. This process does not constitute a proof, however, until it has been established that the steps are reversible.

The analytic approach to the theorem given above would be as follows:

1. By what methods can two angles be proved equal?	1. Among other methods, two angles can be proved equal if they can be shown to be corresponding angles of congruent ▲.
2. Is it possible to introduce two congruent ▲ into the figure?	2. Since $AB = AC$ by hypothesis, I see that by drawing the bisector of $\angle BAC$ I shall have the two congruent ▲ BAD and CAD.

Diagrammatically the patterns of implication in synthetic and analytic argument are as follows (→ is to be read "implies" and ← "is implied by"):

1. *Synthetic.* Hypothesis of unproved proposition → hypothesis of previously proved proposition, axiom, or postulate → conclusion of previously proved proposition, axiom, or postulate → conclusion of unproved proposition.

$$H_1 \rightarrow H_2 \rightarrow C_2 \rightarrow C_1$$

2. *Analytic.* Conclusion of unproved proposition ← conclusion of previously proved proposition, axiom, or postulate ← hypothesis of previously proved proposition, axiom, or postulate ← hypothesis of unproved proposition.

$$C_1 \leftarrow C_2 \leftarrow H_2 \leftarrow H_1$$

Of course, the chain of implications can become much more complex than that indicated here, but the basic pattern remains the same regardless of the degree of complexity.

The analytic process does not constitute a proof until the steps are shown to be retraceable in the reverse order, which, of course, is the synthetic arrangement of the argument. The proof that is most meaningful and provides opportunity for maximum understanding on the part of the pupil is the analytic-synthetic type of argument. The analysis helps to bring out the why of each step taken and the synthesis to establish the rigor of the proof.

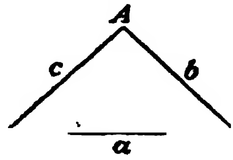

Fig. 26.

As an illustration of the need for investigating the reversibility of the steps taken in the analysis of any proposition, let us examine the following fallacious theorem:

In any isosceles triangle the sum of the two equal sides is always equal to the third side.
Given: $\triangle ABC$ with $b = c$.
To prove: $b + c = a$.
Proof:

Analysis

1. Assume $b + c = a$, then $(b + c)(b - c) = a(b - c)$.	1. Both sides of an equality may be multiplied by the same quantity.
2. $(b + c)(b - c) = a(b - c)$ if $b^2 - c^2 = ab - ac$.	2. Each side of equality is expanded.
3. $b^2 - c^2 = ab - ac$. if $b^2 - ab = c^2 - ac$.	3. $(c^2 - ab)$ has been added to both sides of equality in 2.
4. $b^2 - ab = c^2 - ac$ if $b(b - a) = c(c - a)$.	4. Factor each side of equality.
5. But $b(b - a) = c(c - a)$ since $c(c - a) = c(c - a)$.	5. $b = c$ by hypothesis.

Synthesis

1. $b = c$.	1. Hypothesis.
2. $b - a = c - a$.	2. Equals subtracted from equals give equals.
3. $b(b - a) = c(c - a)$.	3. Equals multiplied by equals give equals.
4. $b^2 - ab = c^2 - ac$.	4. Each side of equality expanded.
5. $b^2 - c^2 = ab - ac$.	5. $ab - c^2$ added to both sides of equality in 4.
6. $(b + c)(b - c) = a(b - c)$.	6. Each side of equality factored.
7. $b + c \neq a$, since division by $b - c$ is impossible.	7. $b = c$ by hypothesis and hence $b - c = 0$.

The synthetic argument thus brings out the step in the analysis that is nonreversible and hence points out the fallacy of the theorem.

In this connection it should be pointed out that another form of the analysis might have indicated the fallacy also. For example, in step 1, if the reasoning had been "$b + c = a$ if

$$(b + c)(b - c) = a(b - c),"$$

then one might have argued "only if $b - c \neq 0$, which is a contradiction of the hypothesis that $b = c$."

Direct and Indirect Proof.[1] Direct argumentation takes place when we try to prove a truth as stated, and it may be either synthetic or analytic in nature. If synthetic, the effort is made to follow the deduction through from the hypothesis to the conclusion as stated in the theorem; if analytic, the effort is made to have the deduction lead from the assumption of the conclusion as a true statement to an implication of the hypothesis. In the synthetic form of a direct proof each step comes as an answer to the question: If this statement is true, then what must necessarily follow as a true statement? In the analytic form, each step comes as an answer to the question: If this statement is to be true, then what statement is sufficient to imply its truth? As has been pointed out above, such a chain of sufficient reasons must be carefully checked for reversibility. For example, if it should be desired to prove $\triangle ABC$ an isosceles triangle, it would be sufficient to show that $\triangle ABC$ is equilateral; but the fact that $\triangle ABC$ could be shown to be nonequilateral would not necessarily mean that it was not isosceles.

Indirect reasoning (or indirect proof) is a method of reaching a desired conclusion through the process of investigation and elimination of *all other mutually exclusive possibilities.* Although teachers of demonstrative geometry have been inclined to neglect this form of argument, it is one of the most powerful and one of the most natural techniques of deduction. One prominent logician has estimated that about half of all our reasoned conclusions are arrived at through the method of indirect reasoning.[2] Another has said that "the process of *reductio ad absurdum* is of the greatest importance. It is the most prominent of all the methods by which men learn those truths of Nature that are unitedly known by the name of Science."[3]

A simple illustration of the technique of indirect reasoning is the following:

[1] The major portion of this section appeared in Charles H. Butler, Indirect Method in Geometry, *School Science and Mathematics*, **39** (1939), 325–336.

[2] W S Jevons, "The Principles of Science" (London: Macmillan Co. Ltd., 1920), p 82.

[3] Alfred Milnes, "Elementary Notions of Logic" (London: W. Swan Sonnenschein and Co , 1884), p. 93.

One evening Mr. Jones was reading when his light went out. He immediately set about to discover the source of the trouble. His first thought was that the current was out all over town but a glance at his neighbor's house, where the lights were burning, eliminated that possible explanation. Next, he thought that the lights were out all over his house, but by trying a near-by lamp he found that it gave light, so another possibility was eliminated. By trying another lamp through the same floor plug, the globe of his lamp in another lamp, and his lamp with a globe known to be good, he finally eliminated all possibilities except the fact that there was something wrong with the wiring in the particular lamp he was using. Further investigation centered on the wiring of the lamp revealed the defect which could then be corrected.

The characteristics of indirect reasoning that are put into play in the illustration are the use of contradictory possibilities and the gradual elimination of those which can be established as inconsistent with conditions which are known or can be shown to exist. First, Mr. Jones assumed (*a*) the current is out all over town, or (*b*) the current is not out all over town. The error of the first of these assumptions was established by a glance at his neighbor's house where the lights were burning. Second, he assumed (*a*) the lights are out all over the house, or (*b*) the lights are not out all over the house. The fact that another light would burn eliminated the first of these two possibilities. Similarly, in each succeeding case he was able to contrast two contradictory assumptions and test them for their validity until he was finally able to locate definitely the source of the trouble. Such pairs of contradictory propositions have the following characteristics:

1. They cannot both be true at the same time.
2. They cannot both be false at the same time.
3. If one of them is false, the other must be true.
4. If one of them is true, the other must be false.

If we wish, then, to prove one of two contradictory propositions true, it is sufficient to prove that the other one is false. This, in summary, is the essence of indirect proof in its simplest and most elemental form. It is based upon two fundamental and complementary laws or postulates of logic which, in turn, rest upon a third principle of logic for their application. These two laws are known as "the law of contradiction" and "the law of the excluded middle." The law of contradiction asserts that a thing cannot both be and not be. The law of the excluded middle asserts that a thing must either be or not be. The above-mentioned characteristics of a pair of contradictory propositions are corollaries to these two postulates of logic and really give us our definition of contradictory propositions. This, of course, is to the

effect that, if there exist two propositions having these characteristics, one of them must be true and the other one must be false.

The application of these two laws depends, as has been said, upon a third postulate of logic which asserts that there are only two ways in which a false conclusion may be reached. Either (1) the reasoning may be incorrect or (2) at least one of the assumptions upon which the reasoning is based may be false. If neither of these conditions exists in a given case, the conclusion which is reached must be correct. If either or both exist, the conclusion may be false. If, then, one reaches a false or inconsistent or contradictory conclusion and if he can be sure that he has reasoned correctly, it must follow that he must have started out with at least one false assumption.

There are at least two points of distinction to be noted between the method of indirect reasoning used by Mr. Jones in detecting the trouble with his floor lamp and the method of indirect proof as applied to propositions in geometry. In the first place, propositions in geometry are generally so stated that one knows at the outset the particular thing which he is required to establish. "*Prove that* under such and such conditions *so and so will be true.*" In the case of the lamp, as in many practical situations, it was not so simple as this at the outset. It was not a case of "prove that the trouble is in the lampstand" but rather, "find out in which of several possible places the trouble is located." In life situations the problems are generally not so strictly defined and delimited as they are in most propositions in geometry. Moreover, it is generally easier in geometric situations than in life situations to be sure that we have included all possibilities in our setup of the problem.

Secondly, the procedure is not identical in the two cases. It is usually more formal in geometry. We set up our pair of contradictory statements, assume as true the one which we wish to prove untrue, and then set about showing, through a chain of logical reasoning, that this supposition necessarily leads to a contradiction or an inconsistency. This is the recognized and well-defined procedure. On the other hand, in a practical situation we rarely set down our possibilities in the manner of formal contradictory statements. We generally proceed more or less intuitively. But even if we should formalize the problem, our investigation of possibilities will often be more in the nature of open-minded experiment than by way of establishing a preconceived opinion by logical reasoning. For example, the light bulb was tested, not with the idea of proving logically that it was defective nor with the idea of proving that it was good, but rather with the idea of

seeing whether or not it was defective. Such experiment often requires less mental effort than the formal proof of the falsity of a geometric proposition, so that, while indirect proof in geometry is generally less complicated in its setup, this advantage is probably more than offset in many cases by the difficulty of the intellectual effort required in the investigation. *This may explain why many people who have difficulty in using formal indirect proof in geometry are able quite successfully to apply more or less informal indirect reasoning in nongeometric situations.*

The technique of indirect proof in geometry may be conveniently analyzed into four very specific steps to be followed:

1. Set up a pair of contradictory propositions, one of which it is desired to prove true. Select this latter one at the outset.
2. Assume (for the time being) that the other one is true, and test the consequences by deductive reasoning to see whether this assumption leads to a contradiction or an inconsistency.
3. If the assumption does lead, by correct reasoning, to an inconsistency or a contradiction, conclude that it was a false hypothesis.
4. Under the above conditions, conclude that the other one of the contradictory propositions (*i.e.*, the one you want to prove true) is necessarily true, since the only alternative proposition has been shown to be false.

One feature of the indirect proof which frequently proves to be troublesome is the clumsy wordiness inseparable from the verbal statement of the denial of the hypothesis in step 3. This hypothesis is generally stated in negative form, and, if the denial of a negatively stated proposition is verbalized, the statement tends often to become very confusing.

A plan that has been used very satisfactorily in overcoming this difficulty is that of writing out separately the complete and careful statements of the two contradictory propositions in step 1 *and then substituting for each of these statements a single identifying symbol.* The Greek letters θ and ϕ were chosen merely because it was felt that they were not likely to be confused with the customary symbols used in identifying points, lines, etc. The choice turned out to be a happy one for another reason, *viz.*, the newness of these symbols aroused the curiosity and interest of the students and, as a by-product of this focusing of attention, the mechanical outline of indirect proof tended to crystallize in the students' minds more quickly and more definitely than had been expected.

The use of these symbols made it possible to eliminate a great deal of the verbal confusion above-mentioned and to shorten both written

and oral exposition, with a corresponding increase in understanding of the mechanics and the nature of indirect proof.

To illustrate, let us consider the proposition:

Two straight lines, both perpendicular to the same straight line, are parallel to each other.

The demonstration following the outline described, but not employing the symbolic representation of the contradictory statements, would be set up in some such manner as the following:

FIG. 27.

Given: $a \perp l$ and $b \perp l$.
To prove: $a \parallel b$.
Proof:

1. Either $a \parallel b$ or a is not $\parallel b$.	1. Contradictory statements.
2. Suppose a is not $\parallel b$.	2. Tentative assumption.
3. If a is not $\parallel b$, then a intersects b.	3. Definition of $\parallel$ lines.
4. In this case we should have two lines both $\perp$ a line from the same point, which is impossible.	4. One and only one line can be drawn from a point perpendicular to a line
5. $\therefore$ It is not true that a is not $\parallel b$.	5. Because this assumption leads to a contradiction of a previously proved theorem.
6. $\therefore a \parallel b$.	6. If one of two contradictory propositions is false, the other one must be true.

This is a perfectly correct and valid proof, but its statement is rather confusing in step 5 where it is necessary to use the double negative or else avoid the use of the word "parallel."

If this proposition were set up in the symbolic form which has been described, the argument could be developed as follows (the same diagram and statement of hypothesis and conclusion may be used):

1. θ . . . $a \parallel b$, ϕ . . . a is not $\parallel b$.	1. Two contradictory statements.
2. Suppose ϕ is true.	2. Tentative assumption.
3. If a is not $\parallel b$, then a intersects b.	3. Definition of $\parallel$ lines.
4. In this case we should have two lines $\perp$ a line from the same point, which is impossible.	4. One and only one line can be drawn from a point perpendicular to a line.
5. $\therefore$ ϕ is not true.	5. Because the assumption led to a contradiction of a previously proved theorem.
6. $\therefore$ θ is true.	6. If one of two contradictory statements is false, the other one must be true.

Observe that the statement "ϕ is not true" is much more concise and less confusing than the verbal statement "it is not true that a is not parallel to b." Moreover, experience has shown beyond doubt that the use of this symbolic representation of the two possibilities in setting up the theorem distinctly increases the students' perception of the essentially contradictory nature of the two statements and clarifies for them the mechanics of the proof.

There are, of course, cases in which the setup of the problem contains more than two possibilities. Take, for example, the proposition:

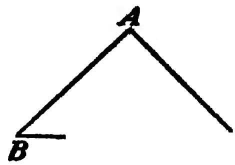

FIG. 28.

If two angles of a triangle are equal, the sides opposite these two angles are also equal.

Given: $\angle B = \angle C$.

To prove: $AB = AC$.

Proof:

1. $\theta \ldots AB = AC$, $\phi \ldots AB \neq AC$, or $\begin{cases} \phi_1 \ldots AB > AC, \\ \phi_2 \ldots AB < AC. \end{cases}$	1. Contradictory statements which involve all possible cases.
2. Suppose ϕ_1 true, then $\angle C > \angle B$. Contradictory to hypothesis.	2. If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side.
3 Suppose ϕ_2 true, than $\angle B > \angle C$. Contradictory to hypothesis.	3. Same reason as step 2.
4. $\therefore \theta$ is true, and $AB = AC$.	4. All other possible relationships between AB and AC have led to contradictions of the hypothesis.

Although the indirect proof is a very powerful instrument in the investigation of truth, it is at times dangerous. One using the indirect type of argument can very easily become guilty of reasoning in a circle, *i.e.*, using the theorem in question either explicitly or implicitly as a reason in its proof. This, however, is but one type of faulty argument which can be the source of error in any form of deductive proof whether direct or indirect, synthetic or analytic. The teacher should be aware of these sources of error and shape his instructional program to insure against them. Probably the most prominent and persistent are the following:

1. Omission of statements
2. Inclusion of irrelevant statements
3. A disregard for a correct order of statements
4. The use of reasons not yet established

5. Reasoning in a circle, *i.e.*, the use of the proposition in question as a reason in its proof
6. The confusion of definitions with theorems
7. The confusion of the hypothesis with conclusion
8. The confusion of a statement with its converse and inverse[1]

Converse, Inverse (Opposite), and Contrapositive. The analysis of geometric truths and proofs by the indirect method frequently demands a familiarity with, and the intelligent use of, the converse, inverse, or contrapositive of some theorem or theorems. An introduction to a clear understanding of these descriptive terms may be secured by examining a simple theorem containing only one hypothesis and one conclusion.

THEOREM. *If a triangle is equilateral, then it is isosceles.* (Obviously true.)

CONVERSE THEOREM: *If a triangle is isosceles, then it is equilateral.* (Not necessarily true.)

INVERSE THEOREM: *If a triangle is not equilateral, then it is not isosceles.* (Not necessarily true.)

CONTRAPOSITIVE THEOREM: *If a triangle is not isosceles, then it is not equilateral.* (Obviously true.)

In this simple case the method of derivation of the different types of theorems is rather evident. For a theorem with one hypothesis and one conclusion:

1. The converse theorem is obtained by interchanging the hypothesis and conclusion.
2. The inverse (or opposite) theorem is obtained by taking the contradiction of the hypothesis as the new hypothesis and the contradiction of the conclusion as the new conclusion.
3. The contrapositive theorem is obtained by taking the contradiction of the conclusion as the new hypothesis and the contradiction of the hypothesis as the new conclusion.

If H represents the hypothesis of a given theorem and C the conclusion, the above definitions may be stated diagrammatically as follows:

THEOREM: *If H is true, then C is true.*

CONVERSE THEOREM: *If C is true, then H is true.*

INVERSE THEOREM: *If H is not true, then C is not true.*

CONTRAPOSITIVE THEOREM: *If C is not true, then H is not true.*

[1] Harry Sitomer, "If-Then" in Plane Geometry, *The Mathematics Teacher*, 31 (1938), 326–329.

The fact that a theorem is true makes no implication about the truth or falsity of the statement in its converse or its inverse. The truth of each must be investigated upon its own merits. The Law of Contraposition, however, states that a theorem and its contrapositive are equivalent, *i.e.*, if one is true, the other is true, and, if one is false, the other is false.[1]

The above definitions must be modified for theorems that involve more than one hypothesis or more than one conclusion. The following definitions have been suggested as satisfactory generalizations:

1. The converse of a theorem may be obtained by interchanging *any* number of conclusions with an *equal number* of hypotheses.[2]

2. An inverse of a proposition having one conclusion may be formed by contradicting one of the hypotheses and the conclusion.[3]

3. A contrapositive of a theorem containing more than one hypothesis and only one conclusion may be obtained by the interchange of the contradictory of one of the hypotheses with the contradictory of the conclusion.[4]

As an illustration of the nature of converse propositions, consider the

THEOREM: *If two right triangles have the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other, the triangles are congruent.*

FIG. 29.

HYPOTHESES	CONCLUSIONS
△ ABC and $A'B'C'$ (Fig. 29).	△ are $\cong$.
H_1 . $\angle C$ and $\angle C'$ are rt. $\angle$s.	C_1 $a = a'$.
H_2 . $c = c'$.	C_2 . $\angle B = \angle B'$.
H_3 . $b = b'$.	C_3 . $\angle A = \angle A'$.

There is a temptation on the part of the immature pupil to consider

[1] N. Lazar, The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry, *The Mathematics Teacher*, **31** (1938), 170.

[2] *Ibid.*, p. 107. It would be better to say, "A converse of a theorem . . ." since the definition provides for more than one converse of a theorem which contains more than one hypothesis and one conclusion.

[3] *Ibid.*, p. 159. Although the concept of "inverse of a proposition" may be extended to theorems with more than one conclusion, the logical difficulties involved seem to make it advisable to restrict the definition to the one given above for geometry in the secondary school.

[4] *Ibid.*, p. 218. The definition given above represents a rewording of the definition given by Lazar in the reference. Contradictory statements are to be interpreted as satisfying the conditions specified on p. 429 (this book). Lazar points out that a less restricted definition may be given for a contrapositive of a theorem (*ibid.*, p. 170), but that logical difficulties make the one cited above more desirable for elementary geometry (*ibid.*, pp. 170, 218).

the statement "the triangles are congruent" as the one conclusion of the theorem. Upon analysis, however, it is seen that there are in fact three conclusions. Symbolically the above theorem may be written

$$(H_1)(H_2)(H_3) \rightarrow (C_1)(C_2)(C_3)^*$$

It is then evident that the following converses can be obtained by the interchange of one hypothesis and one conclusion.

1. $(H_1)(H_2)(C_1) \rightarrow (H_3)(C_2)(C_3)$
2. $(H_1)(C_1)(H_3) \rightarrow (H_2)(C_2)(C_3)$
3. $(C_1)(H_2)(H_3) \rightarrow (H_1)(C_2)(C_3)$
4. $(H_1)(H_2)(C_2) \rightarrow (C_1)(H_3)(C_3)$
5. $(H_1)(C_2)(H_3) \rightarrow (C_1)(H_2)(C_3)$
6. $(C_2)(H_2)(H_3) \rightarrow (C_1)(H_1)(C_3)$
7. $(H_1)(H_2)(C_3) \rightarrow (C_1)(C_2)(H_3)$
8. $(H_1)(C_3)(H_3) \rightarrow (C_1)(C_2)(H_2)$
9. $(C_3)(H_2)(H_3) \rightarrow (C_1)(C_2)(H_1)$

Careful examination of these nine theorems reveals the fact that in each case but one there is a combination of hypotheses sufficient to give congruent triangles. The exception is Theorem 6, in which case the hypothesis would imply the ambiguous case for a triangle. In each theorem, however, which involves H_1 as one of the conclusions there is not necessarily a true statement. Thus, of the nine converse theorems, only six (1, 2, 4, 5, 7, 8) are necessarily true theorems.

Since converse theorems may be obtained by the interchange of two hypotheses with two conclusions and of the three hypotheses with the three conclusions we have the following additional converses of the original theorem:

10. $(H_1)(C_1)(C_2) \rightarrow (H_2)(H_3)(C_3)$
11. $(H_1)(C_1)(C_3) \rightarrow (H_2)(C_2)(H_3)$
12. $(H_1)(C_2)(C_3) \rightarrow (C_1)(H_2)(H_3)$
13. $(C_1)(H_2)(C_2) \rightarrow (H_1)(H_3)(C_3)$
14. $(C_1)(H_2)(C_3) \rightarrow (H_1)(C_2)(H_3)$
15 $(C_2)(H_2)(C_3) \rightarrow (C_1)(H_1)(H_3)$
16. $(C_1)(C_2)(H_3) \rightarrow (H_1)(H_2)(C_3)$
17. $(C_1)(C_3)(H_3) \rightarrow (H_1)(C_2)(H_2)$
18. $(C_2)(C_3)(H_3) \rightarrow (C_1)(H_1)(H_2)$
19. $(C_1)(C_2)(C_3) \rightarrow (H_1)(H_2)(H_3)$

As above it is evident that no theorem which has the clause H_1 as a part of its conclusion is necessarily a true theorem. Furthermore, the hypothesis of Theorem 12, $(H_1)(C_2)(C_3)$, is not sufficient to establish congruency. Hence, of the above converse theorems, only Theorems

* The symbol $\rightarrow$ is to be read "implies."

10 and 11 are necessarily true theorems. Thus out of the total of 19 possible converse theorems to the original true theorem there are only eight (1, 2, 4, 5, 7, 8, 10, 11) which are necessarily true theorems.

For an illustration of inverses and contrapositives of a theorem consider the

THEOREM: *If two triangles have two sides of the one equal, respectively, to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*

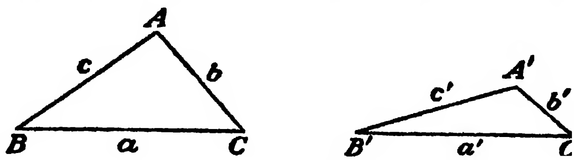

FIG. 30

HYPOTHESES	CONCLUSIO
$\triangle ABC$ and $A'B'C'$.	C $b > b'$
H_1 $a = a'$	
H_2 $c = c'$.	
H_3 $\angle B > \angle B'$.	

This theorem is well adapted to the consideration of inverses and contrapositives as it has only one conclusion In the discussion which follows, we shall use $\bar{H}_1$ to represent the contradiction of H_1, $\bar{H}_2$ of H_2, $\bar{H}_3$ of H_3, and $\bar{C}$ of C Thus $\bar{H}_1$ becomes $a \neq a'$ (or $a \gtrless a'$); $\bar{H}_2$ becomes $c \neq c'$ (or $c \gtrless c'$), $\bar{H}_3$ becomes $\angle B \ngtr \angle B'$ (or $\angle B \leqq \angle B'$); and C becomes $b \ngtr b'$ (or $b \leqq b'$) From the definition of the inverse of a theorem it is evident that there are three inverses to the given theorem They are:

1 *Hypotheses:* $\bar{H}_1$, H_2, H_3; *Conclusion:* $\bar{C}$.
2. *Hypotheses:* H_1, $\bar{H}_2$, H_3; *Conclusion:* $\bar{C}$.
3. *Hypotheses:* H_1, H_2, $\bar{H}_3$, *Conclusion:* $\bar{C}$

Analysis of these three theorems establishes that Theorem 3 is the only true theorem

Similarly, there are three contrapositives of the given theorem. They are

1. *Hypotheses:* $\bar{C}$, H_2, H_3; *Conclusion.* $\bar{H}_1$.
2. *Hypotheses:* H_1, $\bar{C}$, H_3; *Conclusion:* $\bar{H}_2$.
3. *Hypotheses:* H_1, H_2, $\bar{C}$; *Conclusion:* $\bar{H}_3$.

By the contrapositive law all three of these theorems should be true theorems. Careful analysis reveals that this is the case.

Original Exercises. The original exercise is gaining more and more recognition as an effective instructional medium in geometry. This change in emphasis is to be noted in the transition from the text with nothing but definitions, axioms, postulates, and propositions to the modern text in which much use is made of a large list of well-distributed original exercises and incomplete proofs. In fact, the modern philosophy of geometric instruction emphasizes that the pupil who does not have many opportunities for solving original exercises is missing the real opportunity to learn the nature, function, and techniques of demonstrative geometry.

Original exercises may be divided into three major divisions, *viz.*, (1) propositions to be proved through deductive argument, (2) geometric problems to be solved through applications of algebraic or arithmetical techniques, and (3) problems calling for the construction of certain geometric configurations from given elements, using the straightedge and compasses.

The propositions to be proved may be geometrical in content or entirely nonmathematical in nature. In either event the pupil must appreciate the necessity for reading the proposition carefully; for selecting the hypothesis and conclusion; for using a systematic, sound argument; and, in the case of geometric propositions, for using a compact, significant symbolism in connection with a carefully drawn figure. He must be familiar with the advantages and disadvantages of the different forms of proof, he must know how to construct a chain of reasoning and to detect fallacious arguments and unsound hypotheses. Furthermore he must become skilled in detecting those factors of an argument which justify conclusions. As was indicated in the discussion on pages 424 to 428, the analytic-synthetic type of deduction is probably the most effective technique for discovery of unknown proofs of geometric theorems. The indirect form of argument frequently will be found to be very effective in the handling of such original exercises.

In the handling of nonmathematical propositions the emphasis of instruction is largely the same as in the geometrical type of proposition. The major differences will be in the absence of a simplifying symbolism and in the fact that the attention is centered upon the importance of the use of well-defined terms, the soundness and reasonableness of hypotheses, the evaluation of reasoning, the investigation of the evidence behind any conclusion that is drawn, the nature of converses, and the need for the careful search of an argument for unexpressed assumptions. The materials for such problems are found

in advertisements, editorials, propaganda, schoolroom arguments or debates, community projects, etc.

The solving of geometric problems by algebraic and arithmetical techniques implies the use of an appropriate symbolism and a ready familiarity with the fundamental techniques of algebraic and arithmetical analysis. Not only must the pupil be able to translate from geometry into algebraic and arithmetical relationships, but he must be proficient in interpreting algebraic formulas and arithmetical results in terms of geometric figures.

There are four aspects of any construction problem in geometry: (1) Determine how to use the given elements to construct the required figure; (2) construct the figure; (3) prove that the construction is correct, *i.e.*, prove that the constructed figure has the given elements in it, either directly or indirectly; and (4) discuss the possibilities of construction, *i.e.*, under what conditions is the construction possible, and is the construction unique or not, when possible? These four aspects might be more briefly labeled: (1) analysis, (2) construction, (3) proof, and (4) discussion. To illustrate, let us consider the following problem:

To construct a triangle, given the base, the angle opposite the base, and the sum of the other two sides.

Given elements: a, $b + c$, $\angle A$.

To construct: $\triangle ABC$.

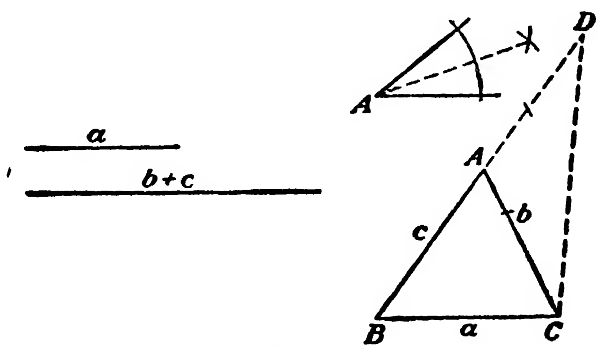

FIG. 31.

Analysis: Suppose $\triangle ABC$ is the required triangle. How can $b + c$ be used to get the triangle? Is there an auxiliary triangle, using $b + c$ and the other given elements, which can be constructed and from which the $\triangle ABC$ can be obtained? It is to be observed that, if BA is extended to D so that $AD = AC$, then $BD = b + c$. Furthermore, since $\triangle ADC$ is isosceles, $\angle D = \frac{1}{2} \angle A$. Therefore $\triangle BDC$ can be constructed since two of its sides and the angle opposite one of them are known. The $\perp$ bisector of DC will then intersect BD in the point A, thus determining the required $\triangle ABC$.

Construction:

On a working line BX lay off $BD = b + c$. At D construct $\angle D = \frac{1}{2}\angle A$. With B as a center and a radius $= a$, describe a circle intersecting DY in C and C'.

Draw BC.

Erect the $\perp$ bisector of DC. It will intersect BD in A.

Draw AC.

$\triangle ABC$ is the required triangle.

Proof:

1. $BC = a$.	1. By construction.
2. $\triangle ADC$ is isosceles.	2. A lies on $\perp$ bisector of DC by construction.
3. $AD = AC$.	3. Legs of isosceles $\triangle$.
4. $\therefore BA + AC = BA + AD$ $= BD$ $= b + c$.	4. $BD = b + c$ by construction.
5. $\angle ADC = \angle ACD$.	5. Base $\measuredangle$ of isosceles $\triangle$.
6. $\angle BAC = \angle ADC + \angle ACD$ $= 2\angle ADC$.	6. Ext. $\angle$ of $\triangle$ = sum of two opposite int. $\measuredangle$.
7. $\therefore \angle BAC = \angle A$.	7. $\angle ADC = \frac{1}{2}\angle A$ by construction.

Discussion: For the construction to be possible, $b + c$ must be greater than a, since the sum of two sides of a triangle must be greater than the third side. In the construction of the auxiliary $\triangle BCD$, the circle with B as a center and a as a radius might intersect the line DY in two points C and C' (as in Fig 32), in which case there would be two solutions for the $\triangle BCD$. If a were equal in length to the $\perp$ distance from B to DY, this circle would be tangent to DY and there would be only one $\triangle BCD$. If a were less than the $\perp$ distance from B to DY, the circle would not intersect the line DY and there would be no $\triangle BCD$ possible From each auxiliary $\triangle BCD$ there can be obtained one and only one required $\triangle ABC$, since the $\perp$ bisector of DC will intersect the line BD in one and only one point. Therefore, there are two solutions, one solution, or no solution for the given problem.

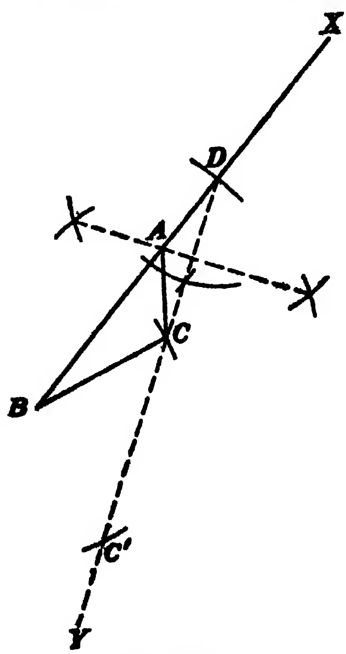

Fig. 32.

No construction problem should be considered a finished product until it has been subjected to the complete treatment outlined above. Much of the most important information concerning a construction is

frequently hidden until the discussion brings it to light. It should be emphasized that such procedure as this gives system to the attack on construction problems and eliminates a great deal of the futile effort expended and feeling of helplessness developed in the "hit-or-miss," "trial-and-error" technique so frequently used.

Concept of Locus. As the analysis of the above construction problem led to the consideration of certain loci, so it is with most construction problems. In the ultimate analysis, a large majority of all constructions depend upon the intersections of loci. In Fig. 32 one locus of the vertex C is the line DY; another locus is the circle with B as a center and a as a radius. The intersection of these loci determines C. Also the perpendicular bisector of DC is the locus of points equidistant from D and C. Hence it locates A on BD. The concept of locus and the methods of construction of loci which satisfy given conditions are fundamental to an intelligent approach to the study of geometric construction.

Conversely, a very effective approach to a clear understanding of locus can be made with the aid of certain elementary constructions. Through the use of such constructions the student can frequently be made aware of the truths of certain locus theorems before he actually comes into formal contact with them. The proper use of experimental geometry in the junior high school provides for just such an approach to the understanding of the locus concept. A few constructions which provide experimental material suitable for this purpose are as follows:

1. Bisect the angles of a given triangle.
2. Draw the perpendicular bisectors of the sides of a given triangle.
3. Draw the altitudes of a given triangle.
4. Draw the medians of a given triangle.

Through the use of these simple constructions, the general experiences of the child, and the use of simple geometric models the introduction of the concept of locus can be made very meaningful to the pupil. Both the dynamic and static concepts of locus should be introduced and developed. The concept of motion provides for the dynamic interpretation of locus as the path of a point moving in such a way that it satisfies certain given conditions. In using the concept of motion, attention should be directed first to the point as it moves under the given restrictions and then to the path it generates. Frequently it is true that this concept is easier to comprehend than is the static interpretation of locus as the place where all points are to be found which satisfy the given conditions. It is, of course, evident that

the two interpretations are in no sense contradictory or mutually exclusive. They are interchangeable in nature; in the one we think of a single point moving to give us the locus, while in the other we think of the locus as a composite picture, as it were, of many points fixed in position. For example,

Dynamic	Static
A point, moving in such a way that it is always equidistant from two given points, generates the perpendicular bisector of the line joining the two points.	The locus of points equidistant from two given points is the perpendicular bisector of the line joining the two points.

It is generally recommended that locus problems be considered in conjunction with the four fundamental theorems on concurrent lines:

1. The altitudes of a triangle meet in a point.
2. The medians of a triangle meet in a point.
3. The bisectors of the angles of a triangle meet in a point.
4. The perpendicular bisectors of the sides of a triangle meet in a point.

These theorems will have been intuitively established through experimentation with the simple constructions mentioned on page 441. They form the basis for two other fundamental locus constructions, *viz.*,

1. Inscribe a circle within a given triangle.
2. Circumscribe a circle about a given triangle.

As an aid to general construction problems there are seven locus theorems which should be considered as fundamental and which should be thoroughly understood by everyone. These seven fundamental locus theorems are:

1. *The locus of a point moving so that it is always at a given distance from a fixed point is a circle with the fixed point as the center and the given distance as the radius.*

2. *The locus of a point moving so that it is always at a given distance from a fixed line is a pair of lines parallel to the fixed line and the given distance from it.*

3. *The locus of a point moving so that it remains equidistant from two fixed points is the perpendicular bisector of the line segment joining the two fixed points.*

4. *The locus of a point moving so that it remains equidistant from a pair of fixed intersecting lines is a pair of lines bisecting the angles formed by the two fixed lines.*

5. *The locus of a point moving so that it remains equidistant from two fixed parallel lines is a line parallel to the two lines and midway between them.*

6. *The locus of the vertex of the right angle of a right triangle with a given fixed hypotenuse is the circumference of the circle with the hypotenuse as diameter.*

7. *The locus of the vertex A of the triangle ABC with fixed base BC and vertex angle A of given magnitude is the arc of a circle on BC as a chord and in which the angle A can be inscribed.*

The traditional method of proving a locus problem has been to prove a theorem and one of its converses. Since the contrapositive of a theorem is true if the theorem is true, this two-way method of proof can also be accomplished by proving one of the contrapositives of the theorem and one of the contrapositives of the converse. The two-way method of proof of any locus problem may then be established by proving two theorems, one each from Group I and Group II, respectively:

Group I—theorem and all its contrapositives
Group II—a converse of the theorem and all the contrapositives of this converse

Such a pairing of theorems exhausts all possibilities for a two-way proof of any locus and provides many choices of method. For example, if a given statement of a locus involves two hypotheses H_1 and H_2, and one conclusion C, then we have

Group I:
1. Theorem: $(H_1)(H_2) \rightarrow (C)$.
2. Contrapositive: $(H_1)(\bar{C}) \rightarrow (\bar{H}_2)$.
3. Contrapositive: $(\bar{C})(H_2) \rightarrow (\bar{H}_1)$.

Group II:
1. Converse: $(H_1)(C) \rightarrow (H_2)$.
2. Contrapositive: $(H_1)(\bar{H}_2) \rightarrow (\bar{C})$.
3. Contrapositive: $(\bar{H}_2)(C) \rightarrow (\bar{H}_1)$.

Thus, since each theorem of Group I may be paired with any theorem of Group II, there are nine ways of giving a two-way proof of a locus problem with two hypotheses and one conclusion.[1] As an illustration, consider the following theorem:

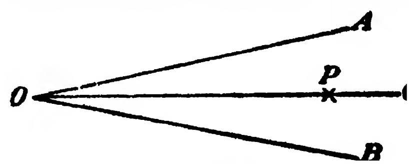

FIG. 33

THEOREM: *The locus of a point equidistant from the sides of a given angle is the bisector of the angle*

[1] Lazar, *op. cit.*, p. 224.

Group I

THEOREM: *If OC is the bisector of ∠AOB and P lies on OC, then P is equidistant from OA and OB.*

CONTRAPOSITIVE: *If OC is the bisector of ∠AOB and P is not equidistant from OA and OB, then P does not lie on OC.*

CONTRAPOSITIVE: *If P is not equidistant from OA and OB and P lies on OC, then OC is not the bisector of ∠AOB.*

Group II

CONVERSE: *If OC is the bisector of ∠AOB and P is equidistant from OA and OB, then P lies on OC.*

CONTRAPOSITIVE: *If OC is the bisector of ∠AOB and P does not lie on OC, then P is not equidistant from OA and OB.*

CONTRAPOSITIVE: *If P does not lie on OC and P is equidistant from OA and OB, then OC is not the bisector of ∠AOB.*

The proof of any theorem from Group I and any theorem from Group II will constitute a two-way proof of the above locus problem. There are nine ways then of establishing the above locus by a two-way proof.

If the seven fundamental locus theorems on pages 442 to 443 are established by the two-way method, or merely postulated, then a one-way proof can be used effectively in practically all other locus situations. In order to avoid the two-way proof some recommend that the seven fundamental loci be postulated. As an illustration of the one-way proof, two locus problems will be considered.

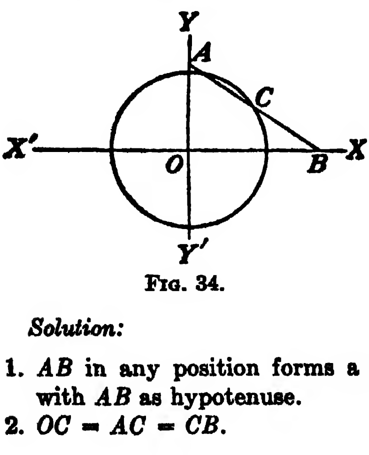

FIG. 34.

Problem. Find the locus of the mid-point of a rod whose ends always touch two fixed rods which are perpendicular to each other.

Given: Rods XX' and YY' ⊥ each other at O. AB a rod moving so that the end A is always on YY', and end B is always on the rod XX'.

To find: The locus of C, the mid-point of AB.

Solution:

1. AB in any position forms a rt. with AB as hypotenuse.	1. XX' and YY' given ⊥ each other.
2. $OC = AC = CB$.	2. The mid-point of the hypotenuse of a rt. △ is equidistant from the three vertices.
3. ∴ The locus of C is a circle with O as center and OC as radius.	3. The locus of a point moving so that it is always at a given distance from a fixed point is a circle with the fixed point as the center and the given distance as the radius. (Theorem 1, page 442.)

Problem. Find the locus of the points of contact of tangents drawn from a given point to concentric circles.

Given: Concentric circles with center at O and tangents drawn from fixed point P.
To find: The locus of the points of contact of these tangents.

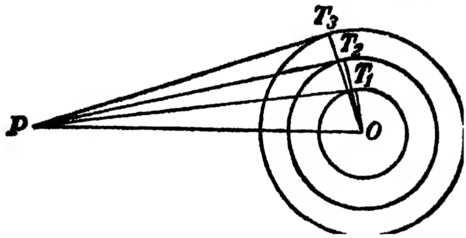

Fig. 35

Solution:

1. In any position OT will pass through O.	1. In each case OT is a radius of a circle.
2. In any position PT will pass through P.	2. Given condition.
3. In any position the $\angle PTO$ is a rt $\angle$.	3. The radius of a circle is always $\perp$ a tangent at the point of contact.
4. $\therefore$ The locus of the point T is a circle on PO as diameter.	4. The locus of the vertex of the rt. $\angle$ of a rt. $\triangle$ with a given fixed hypotenuse is the circumference of the circle with the hypotenuse as diameter. (Theorem 6, page 443.)

Each of the above problems can be established by a two-way proof, but the directness and simplicity of the one-way method make it very desirable.

Another important aspect of the concept of locus is its power as a correlating link between plane and solid geometry. Wilt states that "probably no topic offers richer materials for bringing together the content of plane and solid geometry."[1] She suggests that we state locus problems and teach locus concepts both from the point of view of plane geometry and solid geometry simultaneously, *e.g.*, the locus of a point equidistant from the sides of a plane angle and also from the faces of a dihedral angle; the locus of a point in the plane equidistant from two points and also the locus of a point in space equidistant from two points, etc.

Conversely, it is also suggested that the consideration of some of the more familiar space concepts might lead to a more natural and a better motivated approach to the somewhat abstract concept of locus. The consideration of the parallel walls of a room, the parallelism of

[1] May L. Wilt, Teaching Plane and Solid Geometry Simultaneously, *Fifth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1930), p. 66.

floor and ceiling, or parallel rows of trees in an orchard might lead to a more intelligent comprehension of the true nature of parallelism than the mere drawing of lines on paper or on the board.[1]

The close relationship existing between some of the two- and three-dimensional loci is indicated in the following discussion. The plane locus is stated and then the concept of motion is introduced to develop the space locus.

I. The locus of a point moving in a plane so that it is always at a given distance from a fixed point is a circle with the fixed point as center and the given distance as the radius.

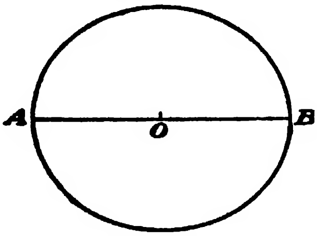

Fig. 36.

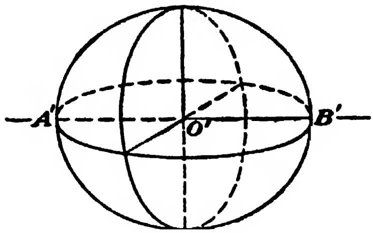

Fig. 37.

If in Fig. 36 the circle is allowed to turn on AB as an axis, we obtain the sphere of Fig. 37 whose diameter $A'B'$ is equal to AB. Every point on the surface of the sphere is equidistant from O' the midpoint of $A'B'$. Hence

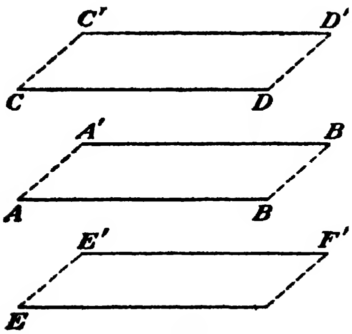

Fig. 38.

1. *The locus of a point moving in space so that it is always at a given distance from a fixed point is a sphere with the fixed point as center and the given distance as the radius.*

II. The locus of a point moving in a plane so that it is always at a given distance from a fixed line is a pair of lines parallel to the fixed line and at the given distance from it.

In Fig. 38 consider first the lines AB, CD, and EF. If the distance between CD and $AB = d =$ distance between EF and AB, then CD and EF are the lines of the plane locus II above. Now, if AB moves

[1] Gertrude E. Allen, An Experiment in the Redistribution of Materials for High School Geometry, *Fifth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1930), pp. 73–74.

to the position $A'B'$ and at the same time CD moves to $C'D'$ and EF to $E'F'$ and they each remain at all times the distance d from AB, then the planes $CDD'C'$ and $EFF'E'$ are each d distance from the plane $ABB'A'$. Hence

2. *The locus of a point moving in space so that it is always at a given distance from a fixed plane is a pair of planes parallel to the fixed plane and at the given distance from it.*

III. The locus of a point moving in a plane so that it remains equidistant from two fixed points is the perpendicular bisector of the line segment joining the two fixed points.

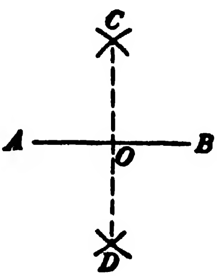

Fig. 39.

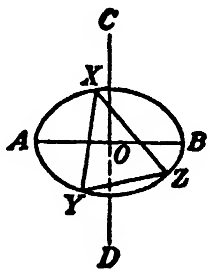

Fig. 40.

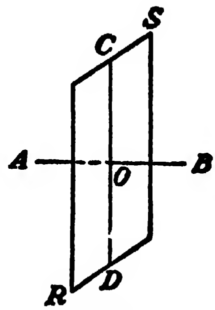

Fig. 41.

If in Fig. 39 the line AB is allowed to revolve around CD as an axis, then Fig. 40 is obtained in which every point on CD is equidistant from the points on the circle XYZ whose center is O and whose radius is $OA = OB$. Furthermore, if C is a fixed point then $CA = CB$. Hence

3. *The locus of a point moving in space equidistant from the points of a circle is a line perpendicular to the plane of the circle at the center of the circle.*

4. *The locus of a point moving in space so that it remains equidistant from the three vertices of a triangle is a line perpendicular to the plane of the triangle at the point which is the center of the circumscribed circle of the triangle.*

5. *The locus of a point moving in a plane so that it is at a given distance from a fixed point not in the plane is a circle whose center is the projection of the point on the plane and whose radius is the projection of the given distance upon the plane.*

If in Fig. 39 the line CD is allowed to revolve around AB as an axis and if it is remembered that CD is unlimited in extent, then Fig. 41 is obtained in which every point in the plane RS is equidistant from the points A and B. Hence

6. *The locus of a point moving in space so that it remains equidistant from two fixed points is a plane perpendicular to the line segment joining the two points and at its mid-point.*

IV. The locus of a point moving in a plane so that it remains equidistant from a pair of fixed intersecting lines is a pair of lines bisecting the angles formed by the two fixed lines.

In Fig. 42 first consider the angles formed by the lines AB and CD with their bisectors EF and GH. Then, if the lines are allowed to move so that the respective planes are generated, it follows that

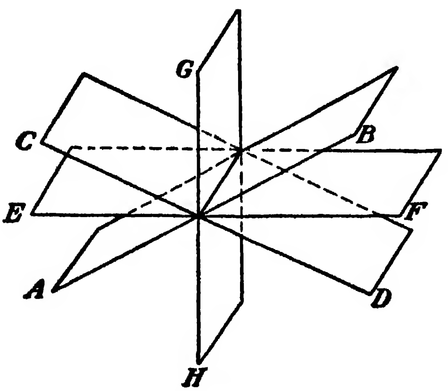

Fig. 42.

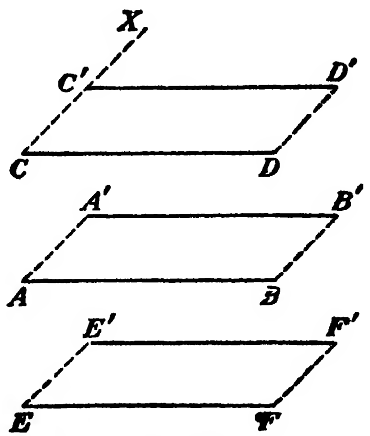

Fig 43.

7. *The locus of a point moving in space so that it remains equidistant from the faces of two intersecting planes is a pair of planes bisecting the dihedral angles formed by the planes.*

V. The locus of a point moving in a plane so that it remains equidistant from two parallel lines is a line parallel to the two lines and midway between them.

In Fig 43 first consider the parallel lines CD and EF with AB midway between and parallel to them. Let the point C move along the line CX while the given lines move in such a way that the initial relationship remains true. Then we have

8. *The locus of a point moving in space so that it remains equidistant from the faces of two parallel planes is a plane parallel to the two planes and midway between them.*

Thus by using motion in space as the generalizing technique the two- and three-dimensional concepts of locus can be shown to be closely associated with each other, and each may be effectively used to supplement the discussion of the other.

The Concept of Dependence.[1] The dynamic concept of locus emphasizes the importance of functional dependence in the analysis of geometric configurations. This concept of the interdependence of geometrical magnitudes is by no means new. Most of the emphasis in the study of functionality on the secondary level, however, has been confined to the analysis of interdependent variation by means of algebraic techniques. In plane geometry this has resulted in a tendency to overlook the fact that there are many aspects of dependence which are not expressible in the conciseness of an algebraic formula but which are deeply significant to a fundamental understanding and appreciation of the real nature of geometrical subject matter.

In 1872 Felix Klein, in outlining his famous Erlangen program, gave the following definition of geometry: "Geometry is the study of the invariants of a configuration under a group of transformations." Thus the structure of geometry may be outlined as follows:

1. *Select the space, i.e.,* determine whether the geometry to be studied is in one, two, three, or higher dimensions.
2. *Select the element, i.e.,* specify the undefined elements. In ordinary two- or three-dimensional geometry the undefined element is generally taken to be either the point or the straight line. This, however, is not necessary.
3. *Build configurations* such as triangles, quadrilaterals, circles, polygons, etc.
4. *Select Transformations.* In elementary geometry, the two most frequently used groups of transformations are those of rigid motion and projection.
5. *Study Invariants.* An invariant of a geometric configuration is a property that does not change in the process of being transformed. For example, length of a line does not change when the line is moved about in space, but it does change if the line is projected from one plane to another; hence length is an invariant under rigid motion but not under projection.[2]

From the above outline of the structure of geometry it is evident that the very nature of the geometry to be studied is dependent upon making certain basic choices. In the geometry of the secondary school the space in which we are interested is either two dimensional (plane) or three dimensional (solid). The undefined element is the point and the transformations are those of rigid motion, *viz.*, rotation and translation. Under these transformations such geometric properties as

[1] Much of the content of this section appeared in F. L. Wren, The Concept of Dependence in the Teaching of Plane Geometry, *The Mathematics Teacher*, **31** (1938), 70–74.

[2] *Cf.* E. P. Lane, Definition and Classification of Geometries, *School Science and Mathematics*, **30** (1930), 50–56

length, distance, size of an angle, united position of point and line, area, etc., are invariant properties. We proceed, then, to analyze the geometric configurations in terms of these invariants, for example:

1. *Two triangles are congruent when their sides are of the same respective lengths.*
2. *Two triangles are congruent when two pairs of sides of the same respective lengths include angles of the same size.*
3. *A circle is the locus of a point moving at a given distance from a fixed point.*
4. *Two circles are congruent if their radii are of the same length.*

Although the point may be taken as the undefined element of the geometry of the secondary school and all other elements defined in terms of it, from a pedagogical point of view, this is very undesirable. It is not good psychology to crowd the young mind with so many formal definitions. No significant mathematical rigor is lost in taking for undefined elements in secondary geometry such terms as point, line, surface, plane, solid, and space. An intelligent comprehension of such concepts can be established intuitively, and it is pedagogically unsound to attempt to build up definitions of these concepts that could be accepted as technically correct.

In any geometric configuration there exist intrinsic interrelations among the constituent elements. An analysis of this interdependence of elements is one of the most effective techniques for discovering the characteristic properties of the configuration and portraying its complete geometric significance. For example,

1. The area of a triangle depends upon the lengths of the altitude and base. What happens to the area when either the altitude or the base is doubled? What happens when both are doubled?
2. How do the circumference and area of the circle depend upon the radius? Which is affected more by a change in the length of the radius?
3. In a cylinder $V = \pi r^2 h$. Which would increase the volume more, to double h or to double r?
4. In a triangle how is a side affected by increasing the opposite angle if the lengths of the two including sides remain constant?

Questions such as these help to bring to light the exact nature of any geometric configuration under investigation. Critical analysis and intelligent interpretation of such configurational dependence will contribute to enriched geometrical comprehension.

The study of geometrical dependence is further enhanced by the principle of continuity which asserts that a proposition which has

been established in relation to a given figure will remain true when that figure changes continuously subject to the conditions controlling its initial construction. The interrelated concepts of dependence and continuity unite to replace a static, mechanical treatment of geometrical subject matter by a dynamic, functional program of instruction. It behooves every teacher of geometry to utilize the full benefits of such an approach to the study of geometrical subject matter. As an illustration of the full significance of the introduction of these dynamic concepts into the teaching of plane geometry, consider the implications as to relational thinking embodied in the two following theorems:

1. *The angle formed by two intersecting lines of unlimited length which meet a circle is measured by one-half the algebraic sum of the intercepted arcs.*
2. *In a triangle the square of the side opposite a given angle is equal to the sum of the squares of the other two sides diminished by twice the algebraic product of one of these sides by its projection upon the other.*

It is true that the concept of directed line and arc lengths must be introduced in the consideration of the above theorems for their full significance. Why should we not use such concepts of directed line lengths in the teaching of plane geometry as a significant aspect of the concept of directed numbers introduced in the algebra of the junior high school? The consideration of other similar groupings of significant theorems will enhance the value of these dynamic concepts of dependence and continuity as instructional mediums in plane geometry.

Since it takes only two points to determine a line, we say that three points are dependent if they are on the same line. This concept of dependence has some very interesting and important applications to the construction of geometric figures. A triangle is uniquely determined by three points not on the same line. In more general terminology this statement would read: *A triangle is uniquely determined by three independent points,* or still more generally, *a triangle is uniquely determined by three independent elements* (*or conditions*). The truth of this last statement is illustrated by the congruency theorems which require three independent elements.

Since the sum of angles of a triangle is always 180 degrees, the three angles are dependent elements. Why are two triangles whose three angles are, respectively, equal not necessarily congruent? Why do three lines through the same point not determine a triangle? Two other less evident illustrations of dependent elements are to be seen in Figs. 44 and 45.

In Fig. 44 it is to be noted that any angle inscribed in the arc BAC will be equal to angle A. Hence it is evident that the elements, a, A, R (radius of circumscribed circle) are dependent elements; *i.e.*, given any two of them the third is determined.

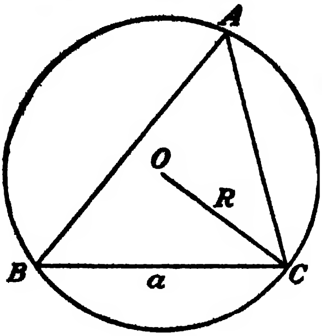

FIG. 44.

In the triangle ABC (Fig. 45) let h_b and h_c represent the altitudes upon the sides b and c, respectively. In the rt. ▲ BFC and BGC, respectively, it is evident that a, h_b, $\angle C$ and a, h_c, $\angle B$, are sets of dependent elements since, in each case, any two of the elements determine the right triangle, and the third element is then uniquely determined. Now let BA be extended to D so that $AD = AC$, then $BD = b + c$. Draw $DE \perp BF$ extended. $DE \parallel AC$; hence $\angle D = \angle CAB$. Also $BE = h_b + h_c$. It then follows immediately from the right triangle BDE that the elements $b + c$, $h_c + h_b$, $\angle A$ are dependent elements

Using both direct and indirect elements related to the unique determination of triangles, five sets of three dependent elements each have been exhibited. They are

I. A, B, C
II. a, A, R
III. $b + c, h_c + h_b, A$
IV. a, h_b, C
V. a, h_c, B

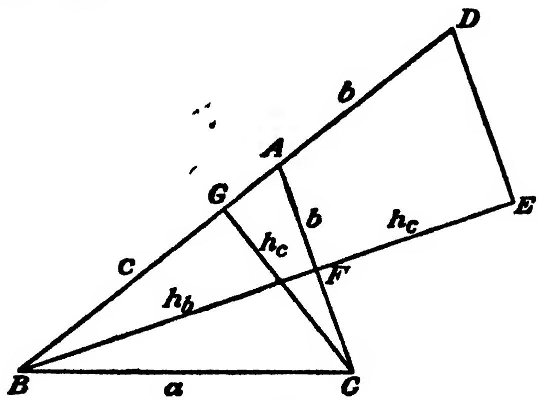

FIG. 45.

Such sets of dependent elements are important aids in an effective approach to exercises in the construction of triangles. It is the purpose of this discussion to call attention to an aspect of this concept of dependence that seems to be generally overlooked in the teaching of plane geometry. To simplify the discussion, two specific examples will be developed in the hope that inferences will lead to the individual development of many others.

A simple construction problem that frequently comes early in the geometry course is:

To construct a triangle having given two angles and the included side.

Let us specify that the given elements are B, C, a. After this construction has been completed, additional construction problems may be derived from it by using the dependent elements of sets I, II, IV, and V. These problems may be used as instructional aids, supplementary drill material, or as material for enriching the study of construction. It would indeed be fine instructional technique to have the students derive and construct such supplementary problems. From set I it is evident that, once angles B and C are known, angle A is also known. Hence the given elements imply the following elements as given:

1. A, B, a
2 A, C, a

From set II it is evident that, since angle A and side a are known, R, the radius of the circumscribed circle, is also known, so that we have

3. A, B, R
4. A, C, R
5. B, C, R
6. R, B, a
7. R, C, a

Now, if we apply sets IV and V, in turn, to the original problem and derived problems 1, 2, 6, and 7, we obtain

8. B, C, h_b
9. h_c, C, a
10. h_b, C, h_c
11. B, C, h_c
12. B, h_b, a
13. h_b, h_c, a
14. B, h_c, h_b
15 A, h_b, a
16 A, B, h_b
17. A, h_c, a
18 A, C, h_c
19. R, B, h_c
20 R, h_b, a
21. R, C, h_b
22. R, h_c, a

Another interpretation of set I is that, if we have given one angle of a triangle, we also know the sum of the other two. Applying this to problems 6, 7, 9, 10, 12, 14, 15, 17, 19, and 21 above, we obtain

23. $R, A + C, a$
24. $R, A + B, a$
25. $h_c, A + B, a$
26. $h_b, A + B, h_c$
27. $A + C, h_b, a$
28. $A + C, h_c, h_b$
29. $B + C, h_b, a$
30. $B + C, h_c, a$
31. $R, A + C, h_c$
32. $R, A + B, h_b$

Hence from the original set of elements there have been derived 32 additional sets of given elements with which triangles may be constructed. Each of these problems may have an individual construc-

tion which is independent of the original problem B, C, a. It is quite evident, however, that all of them may be reduced to the original problem. Still others may be derived.

Another illustration of interest may be found in the application of these techniques to derive additional construction problems from the one whose construction was carried out on pages 439 to 440.

From the use of set I we have the following

1. $a,\ B + C,\ b + c$

An application of set II to the original problem and to the derived problem gives

2. $R,\ A,\ b + c$
3. $a,\ R,\ b + c$
4. $R,\ B + C,\ b + c$

An application of set III then gives

5. $a,\ h_c + h_b,\ b + c$
6. $a,\ A,\ h_c + h_b$
7. $R,\ h_c + h_b,\ b + c$
8. $R,\ A,\ h_c + h_b$

Applications of sets I and II to problems 6 and 8 give

9. $a,\ B + C,\ h_c + h_b$
10. $a,\ R,\ h_c + h_b$
11. $R,\ B + C,\ h_c + h_b$

If it is furthermore recalled that the perimeter $2p$ is the sum of the sides, *i.e.*,

$$2p = a + b + c$$

the following 12 sets may be derived:

12. $2p,\ A,\ a$
13. $2p,\ R,\ a$
14. $2p,\ A,\ R$
15. $2p,\ b + c,\ R$
16. $2p,\ B + C,\ a$
17. $2p,\ B + C,\ b + c$
18. $2p,\ b + c,\ A$
19. $2p,\ R,\ B + C$
20. $2p,\ h_c + h_b,\ a$
21. $2p,\ h_c + h_b,\ A$
22. $2p,\ h_c + h_b,\ B + C$
23. $2p,\ h_c + h_b,\ b + c$

Thus 23 additional construction problems have been derived from the original one through successive applications of the concept of geometric dependence.

Similar possibilities exist in all construction situations in geometry. Such instructional technique not only provides the teacher with an enriched program of geometric teaching but also affords the student the opportunity for a more significant understanding of the real nature of geometric construction.

Exercises

1. What is meant by postulational thinking?

2. Select an editorial, news item, or advertisement, and analyze it for terms that need to be defined, assumptions that are made, and conclusions that are drawn or implied. Evaluate the validity and reliability of these conclusions.

3. In your own words distinguish between induction and deduction; synthesis and analysis.

4. Define and illustrate mathematical induction as a form of proof. Is it a form of pure induction?

5. Frequently we hear the expression "synthetic geometry" as contrasted with "analytic geometry." What are the implications as to technique in this contrast between synthesis and analysis?

6. What do you consider the major distinguishing characteristics between direct and indirect proof?

7. Select a problem (construction or proof), and show how the indirect form of argument may be used as an effective instrument in the analysis of the problem.

8. What is meant by the statement that demonstrative geometry is primarily a deductive science?

9. Why is it necessary that, in a deductive science, there must be a list of undefined terms and accepted assumptions?

10. Is there any significant difference in the logical and pedagogical implications of the fact that such lists of undefined terms and accepted assumptions must exist?

11. Distinguish between the implications of "self-evident truth" and "accepted truth."

12. Select a nongeometrical problem, similar to the one on page 429, and show, in detail, how it might be solved by the process of indirect reasoning.

13. Given the theorem: *In equal circles, equal chords are equidistant from the center.*

a. What are the hypotheses and conclusions?

b. Write all converse, inverse, and contrapositive theorems. (Use these terms as defined in this chapter.)

c. Which are true theorems?

14. Show that all the theorems relating to congruence of triangles can be derived by taking converses of any one of them. What untrue theorems are also derived?

15. Give both a static and a dynamic statement of each of the following two-dimensional locus concepts: circle, bisector of an angle, two parallel lines equidistant from a fixed line, ellipse, hyperbola, and parabola.

16. Give an extension to three-dimensional space of each of the locus concepts of exercise 15.

17. In the construction problem: *To construct a triangle, given the elements a, R (radius of circumscribed circle), and h_b (altitude on the side b);* show the analysis, construction, proof, and discussion.

18. Demonstrate the analytic-synthetic type of proof in proving the following theorem: *If ABC is an equilateral triangle inscribed in a circle and P is any point on the arc BC, then $PA = PB + PC$.*

19. What theorems of plane geometry are incorporated in Theorem 1 on page 451; in Theorem 2?

20. Select a group of related theorems from plane geometry, and show how the concept of dependence and the principle of continuity permit the incorporation of all theorems of the group into one theorem.

21. Of the following sets of three elements of a triangle, which are sets of dependent elements?

1. a, b, c	6. a, h_b, h_c
2. a, A, R	7. a, h_b, C
3. A, B, C	8. A, a, h_a
4. A, b, C	9. b, R, h_c
5. A, h_c, b	10. A, R, b

22. Using sets of dependent elements, how many new construction problems can you derive from the three elements given in exercise 17?

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CHAPTER XVII

THE TEACHING OF TRIGONOMETRY AND LOGARITHMS

In the minds of many people trigonometry is associated only with the idea of college mathematics and engineering. This is doubtless true because trigonometry has never been offered in the high schools to the extent that algebra and geometry have, whereas it is universally offered and often required in college because it is indispensable to the study of analytic geometry, calculus, and the advanced courses in mathematics, as well as in the closely allied fields of engineering and the physical sciences. The result of the meager offering of trigonometry in the high schools has been that comparatively few people have really become acquainted with the subject. This, in turn, has caused it to gain a reputation of extreme difficulty which, again in turn, has inhibited the demand for it in the high schools

The advent of the junior high school and the concomitant movement toward the reorganization of secondary-school mathematics, which has been prominent in the past four decades, has had some effect in dispelling this erroneously restricted view of the subject. There has come about the realization that, while certain parts of trigonometry are indeed difficult and suitable only for relatively mature students, there is much, on the other hand, which is quite simple and easily understood even by normal students of junior-high-school age. Indeed it has been found that some of the elementary concepts of trigonometry are more easily comprehended than much of the usual work in algebra and that the application of these principles and relationships are very interesting to children in the early years of the secondary school. Moreover, such application offers an excellent means of correlating arithmetic with certain parts of informal geometry and with the solution of simple linear equations. In addition, the early introduction and simple treatment of these elements of trigonometry may be expected to have considerable influence in counteracting the reluctance of students to undertake later the systematic study of the subject. Therefore, because of its general educational value and its motivating force, some preliminary work in trigonometry has come to be commonly offered in the junior high school. Many

textbooks in ninth-grade algebra or general mathematics include a unit on numerical trigonometry, and not infrequently some of this work is given in the eighth grade.

The general maturity and mathematical experience of junior-high-school students is, of course, inadequate for a systematic study of trigonometry. Here the objectives are different, the subject matter more involved, the analyses more difficult, and the methods more rigorous. Consequently such a study of the subject is properly deferred until the latter years of the senior high school or the first year of college work.

Trigonometry in the Junior High School. Since it seems necessary to defer the systematic study of trigonometry until the latter years of the senior high school, those parts which are to be included in the junior-high-school work must find their justification in the direct contribution which they can make to the goals of general education at this level. These may be thought of as potential practical values[1] and as general educational values. By comparison, the latter are by far the more important. They make for general enrichment of the course and include extension and clarification of the concept of ratio and of the use of symbolism; appreciation of the power and application of indirect measurement; understanding of the methods of accomplishing such measurement through the use of the right triangle and the tangent, sine, and cosine ratios; correlation of ideas and procedures drawn from arithmetic, algebra, and geometry; and stimulation of interest in mathematics as a whole.

Broadly speaking, the teaching of trigonometry should begin long before trigonometric ratios are considered at all. Since these ratios imply measurement, comparison, and the study of certain properties of similar figures, these activities form a potential groundwork. If properly taught, they pave the way for future work by developing concepts which later are to provide the very basis of numerical trigonometry.

The subject matter of the numerical trigonometry of the junior high school is simple. It consists primarily of instances dealing with the indirect measurement of distances. Much impetus can be given to this work by allowing the students to undertake actual field projects, but, before this is done, a considerable amount of preliminary groundwork should be laid. Presumably the students will already be familiar, through their work in informal geometry, with the principles of draw-

[1] The term "practical" is used here in a narrow sense. Under a more liberal interpretation all genuine educational values may be regarded as practical.

ing to scale and with the use of the protractor and tape for measuring angles and distances directly. This is about all that can be assumed safely. It is necessary, however, that they have clear concepts of the meaning of similar figures (especially similar triangles); of a ratio as a comparison of two quantities in the sense that one is a certain fraction of the other or that one is a certain number of times the other; and also of a ratio as a single number (fractional or integral) which may be used as a multiplier. They also need to understand the particular meanings of the tangent ratio, the sine ratio, and the cosine ratio, and they must know how to use the table of natural functions either to find the value of a particular function of a given angle or to find the value of an angle if the numerical value of one of its functions is known. They need to know the meaning of such terms as "angle of elevation" and "angle of depression" They need practice in analyzing problem situations, in making working drawings, in selecting the appropriate functions to use, in setting up and solving equations involving these functions, and in substitution and evaluation These things need to be carefully and clearly explained by the teacher. In order to develop comprehensive understanding of the techniques used in field projects, the students should work a large number of illustrative problems. Furthermore, the teacher should carefully discuss the full implications of such problems with the students.

Developing the Meaning of a Trigonometric Function. In beginning the study of numerical trigonometry the first thing to be done is to make clear the meaning of the trigonometric functions as ratios and as numbers Probably the best way to accomplish this is to have the students actually make careful measurements of the angles and sides of right triangles and compute the numerical values of the ratios representing the tangent, sine, and cosine. It is well to have several students compute the values of these functions for angles of a given size so that they may compare their results. Generally, their results will show a fairly close correspondence, and the fact that they may not agree exactly offers a good opportunity to emphasize the approximate nature of measurement. Thus, any discrepancies may be attributed either to mistakes in computation or to errors in taking the measurements. This approach, through measurement and computation, to the meaning of a trigonometric function emphasizes the concept of the function both as a comparison of the lengths of two sides of a right triangle and as a single numerical quantity or quotient. Common agreement on the value to be accepted may be reached by averaging the values found by several students.

A variation of the foregoing procedure may be made by having each student make several sets of measurements and computations leading to the determination of the values of the functions of a particular angle, such as, *e.g.*, angle A in Fig. 46. Thus the value of the tangent of angle A might be determined using several different ratios such as CB/AB, DE/AE, FG/AG, HK/AK, LM/AM, etc. This would give emphasis to the principle that the value of a given function of any angle is independent of the actual lengths of the sides of the triangle but depends only on the *ratio* of the lengths of the two sides involved.

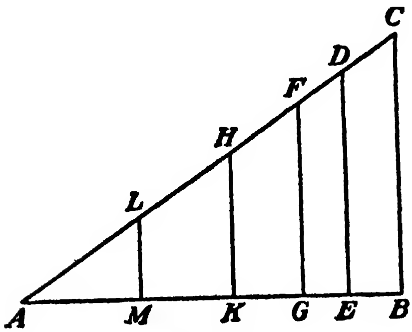


Fig. 46.

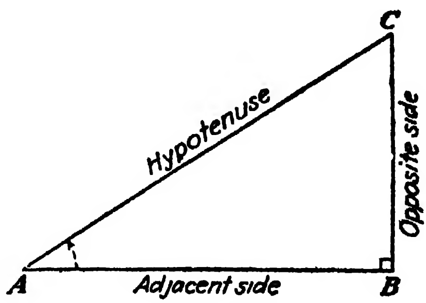


Fig. 47.

The functions that are introduced in the study of trigonometry in the junior high school are generally limited to the tangent, sine and cosine. The inclusion of others would add nothing toward making clear the meaning of a function, and they are not needed for the solution of the simple applied problems that make up the work of this period. For obvious reasons this work is limited to situations that involve functions of acute angles With this in mind, the functions of an acute angle may be defined in terms of the sides of a right triangle containing that angle. Let angle BAC be the given angle (Fig. 47).

$$\text{Tangent } \angle BAC = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\text{Sine } \angle BAC = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\text{Cosine } \angle BAC = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

An interesting and valuable exercise can now be introduced by having the class as a whole make measurements and computations necessary for the compilation of a table of sines, cosines, and tangents for the acute angles, say, which are integral multiples of 5 degrees.

This exercise would provide well-motivated practice in accurate drawing and measurement of lines and angles and in careful computation. Diagrams for the measurements should be rather large and should be very carefully drawn. For each of the angles considered, at least two or three students should make determinations of the functions for purposes of comparison, checking, and averaging of the results. The values of the functions as shown by the computed results should be expressed to three significant figures.

Teaching Students How to Use the Trigonometric Functions. The meanings of the trigonometric functions and the ways in which they are to be used will be more quickly and adequately apprehended if these meanings and uses are illustrated in problem situations. There need be no delay about this. As soon as the meaning of the tangent ratio has been explained to the point of understanding, the teacher should show how it is used in finding distances or angles without direct measurement. Hypothetical or "made-up" problems will serve for this purpose quite as well as "real" problems and perhaps better, because the assumed elements (an angle and a distance or two given distances) can be selected at will and with a view to convenience and there will be no extraneous details of actual measurement, of the manipulation of instruments, or of unnecessarily difficult computations to draw the students' attention from the basic principles and procedures involved. Substantial mastery of the basic theory and procedure should be assured before the students are thrown into situations where they will have to provide their own data. Initial interest is usually high, and the artificial motivation afforded by field projects is unnecessary at the outset.

Thus such problems as the following serve well to introduce students to the applications of the functions. The fact that convenient data are arbitrarily chosen in no way lessens the value of the problems.

Example. From a point 50 feet from the base of a tree, the angle of elevation (A) of the top of the tree is found to be 29 degrees. Find the height of the tree.

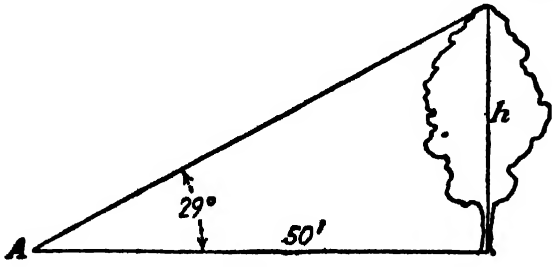


Fig. 48.

The steps in the solution of the problem should be explained to the students about as follows:

1. First we shall make a picture or diagram to represent the problem. On this diagram we shall indicate all data that are given, such as the distance from point A to the base of the tree and the size of the angle of elevation at A. (The diagram is made and data are indicated as shown.)
2. Since the height of the tree represents the unknown distance, we should designate it by some letter, *e.g.*, let us use h.
3. We know that the ratio $h/50$ represents the tangent of angle A, so we may now write the equation $h/50$ = tangent of 29 degrees.
4. We know that the tangent of an angle can be expressed as a number. From our table of tangents we find that the numerical value of the tangent of 29 degrees is 0.5543. Therefore we may substitute this numerical value for the expression "tangent of 29 degrees" and write the equation $h/50 = 0.5543$.
5. Now, if we solve this equation for h, we shall get the equation $50(h)/50 = (0.554)(50)$ or $h = 27.72$ feet, *i.e.*, about 28 feet. Thus we know that the tree is about 28 feet high.

In presenting this explanation, the students' attention should be deliberately focused both upon the particular activities and the basic concepts presented in each of the various steps, and upon the order in which the steps are taken. The order reveals to the students a pattern for their work. This pattern not only helps them to systematize their written work and their computations, but it also helps them in analyzing such problems and in organizing their thinking about them. Each step stresses one important element in the analysis and solution of the problem. Drawing and lettering the figure and indicating the given data (step 1) give the problem a concrete setting and facilitate the job of translating it into an equation. The selection and indication of a literal symbol to represent the unknown part of the figure (step 2) direct attention to the fact that the object of the work is to determine the magnitude of this particular part. Writing the equation (step 3) requires analysis of the problem to determine which of the trigonometric functions is the appropriate one to use.

In this connection the following points should be stressed: (*a*) if one side and an angle of the triangle are given and it is required to find another side, that function should be selected which is represented by the ratio that involves both the unknown side and the known side; (*b*) if two sides are given and an angle required, then that function should be selected which is represented by the ratio of one of these known sides to the other. A scheme that is slightly mechanical yet rather helpful in selecting the proper function is to emphasize that the

sine and cosine should be used if one of the given sides is the hypotenuse. It should further be emphasized that the hypotenuse occurs in the denominator of each of these functions.

The transition from the ratio concept to the numerical concept of a trigonometric function and the substitution of the numerical value for the ratio (step 4) are of vital importance in understanding the use of the functions in indirect measurement. The actual solution of the equation for the unknown part, and the reinterpretation of this in terms of the diagram or of the original problem situation (step 5) brings a realization of how the laws of algebra operate to give the required information by giving explicit form to a relationship which was merely implicit before. In the calculations involved care should be taken to observe the rules for approximate computation. The order in which these steps have been indicated is the order in which they logically occur in the analysis and solution of the problem. Fortunately there is no conflict between this logical order and the natural or "psychological organization" of the analysis from the standpoint of the immature student.

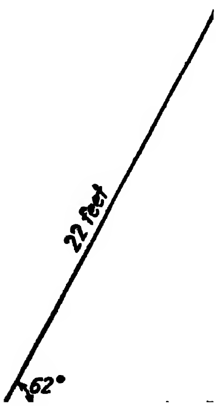


Fig. 49.

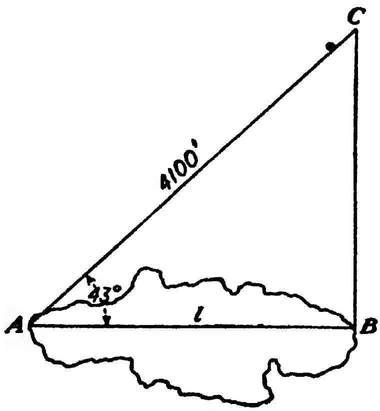


Fig. 50.

The problem upon which the foregoing discussion has been based involves the use of the tangent. Similar illustrative problems involving the sine and the cosine should also be used. Such problems should be selected or devised with care, but they are available in countless numbers and with many variations. The following are examples involving, respectively, the sine and cosine:

Example. A ladder 22 feet long is placed against a vertical wall so that it makes an angle of 62 degrees with the ground. At what height above the ground does the ladder touch the wall (Fig. 49)?

Example. To find the distance across a lake some surveyors sighted an east-west line AB across the lake and then a north-south line BC. Then they measured the distance AC and the angle BAC. They found that $AC = 4{,}100$ feet and that angle $BAC = 43$ degrees. Find how far it was from A to B (Fig. 50).

In setting up the earlier problems care should be taken to arrange the data so that, when the equations are set up, the symbols for the unknown parts will occur in the *numerators* of the fractions representing the trigonometric ratios or functions to be used. Later on it will be desirable to introduce some problems in which the unknown will occur in the *denominator* of the fraction. The following example will illustrate this:

Example. In order to find the distance between two points P and Q on opposite sides of a small lake, two boy scouts decided to set up a right triangle with PQ as the hypotenuse (Fig. 51). They used an angle mirror to locate a point O, such

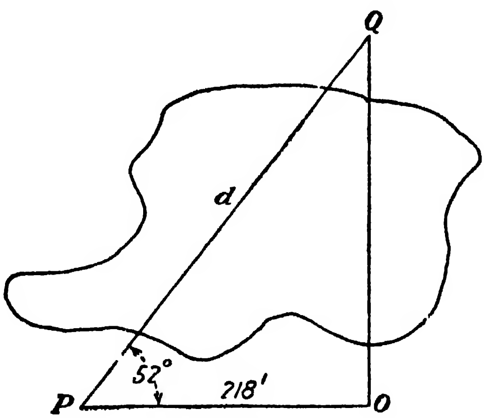


FIG. 51.

that the lines OP and OQ formed a right angle. The distance from O to Q could not be measured directly, but OP was measured and found to be 218 feet. By sighting from P to O and then from P to Q, the size of angle P was established as 52 degrees. From these data the boys found the distance from P to Q. How far was it? (Call this distance d.)

This problem leads to the equation

$$\frac{218}{d} = \cos 52° = 0.6157.$$

To simplify the solution, it may be suggested to the students that this can be written $218/d = 0.616/1$, that by inverting both of these

fractions the equation $d/218 = 1/0.616$ will result, and that, if both members are now multiplied by 218, the equation is solved for d, the required distance; its value may be obtained by performing the indicated division.

The work in trigonometry in the junior high school will scarcely go beyond these simple applications of the functions to the general problem of finding certain unknown parts of right triangles. If desired, the cotangent may be introduced, but it is unnecessary and little is to be gained by it. The secant and cosecant should not be discussed.

The statement made on page 462 with reference to hypothetical problems should not be interpreted as a disparagement of field work. On the contrary, it is certain that appropriate field projects can do much to enhance the interest in the application of trigonometric methods. Students can take a plane table, field protractor, and clinometer and use them in actually performing simple surveying problems. Other instruments such as the tape, angle mirror, plane table, and alidade will also find use in connection with field problems leading to trigonometric solutions. Field work involving the actual measurement of distances and angles often helps to stimulate interest in numerical trigonometry, and projects of this nature may be used to advantage when circumstances permit. Students are generally interested in such projects, but their interest in the activities themselves should not be allowed to obscure their purpose, which is to secure actual data with which to work, nor should the securing of such data involve an inordinate amount of time. If such field work is to yield maximum benefit, it must be carefully planned and supervised by the teacher.

Trigonometry in the Senior High School or in the Junior College. The systematic course in trigonometry in the senior high school or the junior college will be in marked contrast to the trigonometric work offered in the junior high school, although the contrast is not so much in kind as in degree. The added maturity of the students, the enhanced background of algebraic technique, and the selective effect of the work of the tenth and eleventh grades practically ensure an average level of ability and readiness far beyond that which may be expected in the junior high school. Consequently the work in trigonometry at the more advanced level will be characterized by a corresponding increase in completeness, rigor, abstractness, and generality. The students will have to accept more responsibility than before for the analysis of problems and for the mastery of a vastly extended theory of the subject.

This does not mean, however, that the teacher is to be relieved of all responsibility. On the contrary, there are many parts of trigonometry which many students would find inordinately difficult and time consuming to "dig out for themselves" but which can be made clear and meaningful easily by a skillful teacher. The teacher of trigonometry has the same obligation to help his class to a clear understanding of the basic principles and techniques of the subject as has the teacher of arithmetic, algebra, or geometry in the earlier years of the secondary school. With this in mind, the remainder of this chapter will be devoted to a discussion of teaching problems arising in connection with the development of certain concepts and procedures in trigonometry. The topics which will be discussed have been selected on the basis of three criteria:

1. They are fundamental.
2. They are often troublesome.
3. They are based upon relatively simple fundamental principles which, if not properly emphasized, are easily obscured by the mass of detail attending their development but which, if set forth and perceived at the outset, will furnish a clear and helpful guide to the development.

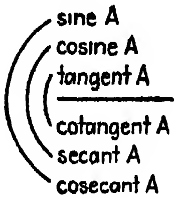
In the discussion of these topics the aim will be to present suggestions for developing them in a manner calculated to make clear the underlying principles and thus to provide a basic framework around which the details of the development may be organized effectively.

Learning the Trigonometric Functions. The first thing which a student must do in beginning the study of trigonometry is to get a clear understanding of the meanings of the trigonometric functions and to learn their names. Suggestions for developing the meanings of the tangent, sine, and cosine of an acute angle have been given earlier in this chapter. These suggestions, made for the teaching of junior-high-school students, are equally applicable to senior-high-school or junior-college students. The latter, however, must learn the meanings of the cotangent, secant, and cosecant. It may be pointed out that the functions are, after all, merely definitions of certain ratios, and the students, having learned the meanings of the sine, cosine, and tangent, will have no difficulty in understanding the meanings of the other functions in terms of the sides of a right triangle.

The student will perhaps be curious to know why the names of three of the six functions are the same as the names of the other three except for the prefix "co." The teacher should explain that the *co*-sine of an angle is the *sine of the complementary angle which is*

the other acute angle in the right triangle, and that the word "cosine" is merely a contraction and combination of the words "complement's sine." Similar explanations can be made of the corresponding relation of the *co*-tangent to the tangent of an angle, and of the *co*-secant to the secant.

In case the students have any difficulty in remembering which ratios are associated with the different names, the association of the cosecant, secant, and cotangent of an angle with the sine, cosine, and tangent of the angle can be made easy by having the students learn to recite the names of the six functions in the usual order, *i.e.*, sine, cosine, tangent, cotangent, secant, and cosecant. If these names are written then in a vertical column as in the diagram, it is apparent that the two functions joined by any one of the three arcs are reciprocals of each other. Thus any one of the last three can be recalled immediately by making this association with the appropriate one of the first three. This mnemonic device has two advantages. It cuts in half the amount of direct memorizing to be done at the outset, and it specifies and emphasizes the reciprocal relationships existing among the functions.



Difficulties connected with the use of the inverse functions are generally due to the fact that their meanings have not been adequately explained. Also the notation $\sin^{-1} \frac{1}{2}$, $\tan^{-1} .3802$, etc., is sometimes confusing in the beginning because there is a very definite tendency to associate this notation with the idea of an exponent. Indeed some writers hold that this notation should be entirely abandoned in favor of the expressions *arcsin, arctan,* etc. While there would probably be a material advantage in this, if it were universally adopted, the fact remains that both forms of notation are likely to persist in textbooks and other published works, so that students should be made familiar with both forms as a matter of practical necessity. Great care should be taken to ensure that the students will realize that the symbols $\sin^{-1} .5$, arctan .8325, etc., *represent angles rather than functions of angles.* It should be emphasized that the symbols for the inverse functions are to be read "the angle whose . . . is"; for example, the two above-mentioned inverse functions should be read "the angle whose sine is .5" and "the angle whose tangent is .8325," respectively. The mastery of concepts of the functions and the inverse functions and of the distinction between them will be facilitated if the students are given systematic drills in writing down given functions and the corresponding inverse functions in parallel columns, thus:

$$\begin{array}{rlrl} \sin 18°28' &= .31675 & \arcsin .31675 &= 18°28' \\ \cot 55°9' &= 69631 & \cot^{-1} .69631 &= 55°9' \\ \cos 34°10' &= .82741 & \cos^{-1} .82741 &= 34°10' \end{array}$$

etc.

Not much time will be required for this. Indeed the drills may be very short, perhaps not more than 3 or 4 minutes at a time, but they should be given at recurring intervals throughout the course.

Teaching the Functions of Special Angles. The special angles 0, 30, 45, 60, and 90 degrees are of great importance and occur so frequently that students should be able to write down the functions of these angles without having recourse to tables of functions. Some teachers prefer to have their students memorize the values of the functions of these special angles. It is possible to do this, but it is not necessary, because they may all be immediately and easily derived from three elementary geometric considerations.

Consider the right triangle ABC with an acute angle A of 45 degrees and a right angle at B. From the illustration we see at once that such a triangle is formed when a diagonal of a square is drawn. If the length of one side of the square is taken as one unit, then each leg of the triangle will be 1 unit in length, and the diagonal of the square, which is the hypotenuse of the triangle, will be found by the Pythagorean theorem to have a length of $\sqrt{2}$ units. With these numbers known, all the functions of 45 degrees can be written down immediately. The only quantities needed are the numbers 1, 1, and $\sqrt{2}$, and even these need not be memorized since they are immediately apparent from the fact that the triangle is formed by a diagonal and two adjacent sides of a square. The results are as follows:

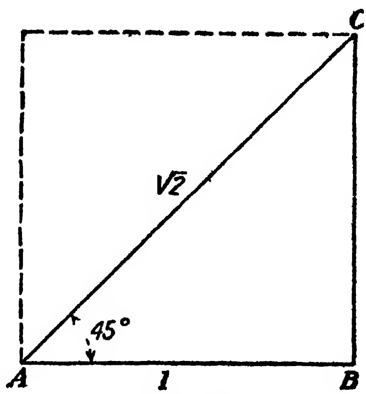


FIG 52

$$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}; \qquad \tan 45° = \cot 45° = 1$$

$$\csc 45° = \sec 45° = \sqrt{2}$$

Consider now an equilateral triangle ADC (Fig. 53) with angle A bisected as shown. The bisector AB forms with AC an angle of 30

degrees and with BC a right angle. Thus we have a 30–60-degree right triangle ABC whose hypotenuse AC is twice as long as the leg BC (since $BC = \frac{1}{2}DC$ and $AC = DC$). If we now consider the length BC as one unit, then AC has a length of 2 units. Again, using the Pythagorean theorem, we get $AB = \sqrt{3}$ units. From Fig. 53 we have

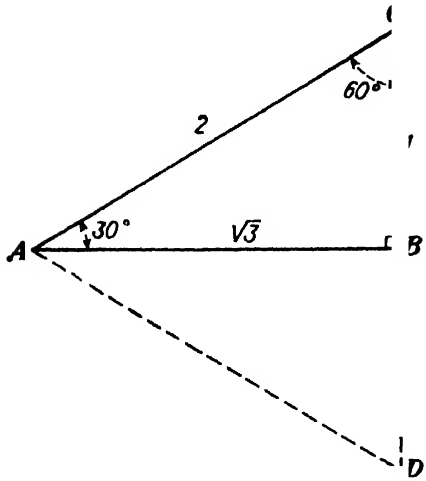


Fig. 53.

$$\begin{aligned}
\sin 30° &= \tfrac{1}{2} = \cos 60° \\
\cos 30° &= \sqrt{3}/2 = \sin 60° \\
\tan 30° &= 1/\sqrt{3} = \cot 60° \\
\cot 30° &= \sqrt{3} = \tan 60° \\
\sec 30° &= 2/\sqrt{3} = \csc 60° \\
\csc 30° &= 2 = \sec 60°
\end{aligned}$$

A most effective scheme for studying the functions of 0 and 90 degrees is the unit circle. Previous discussion has pointed out the fact that ratios of the sides, not the lengths of the sides, determine the values of the different trigonometric functions. Hence, for convenience, the angle of reference may be made the central angle of a circle of unit radius. In Fig. 54

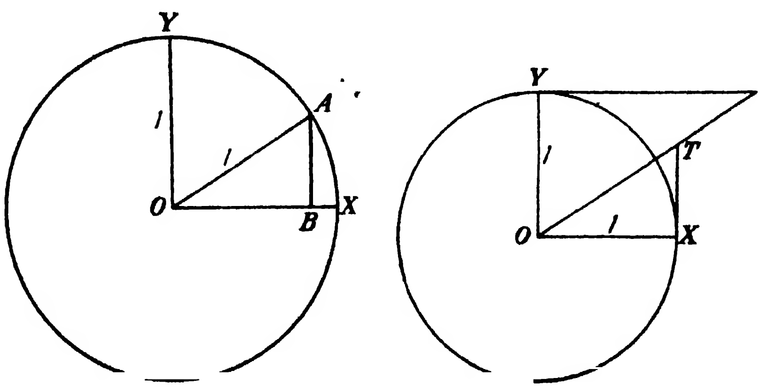


Fig. 54.

$$\sin \angle BOA = \frac{AB}{OA} = \frac{AB}{1} = AB$$

We thus see that we may study the change in the sin $\angle BOA$ by examining the change in AB as $\angle BOA$ increases from 0 to 90 degrees.

When $\angle BOA = 0$ degrees, OA coincides with OX and $AB = 0$; when $\angle BOA = 90$ degrees, OA and AB both coincide with OY. Thus $AB = 1$ when $\angle BOA = 90$ degrees. Incidentally, this discussion can also be used to give significance to the use of the ratio

$$\frac{\text{opposite side}}{\text{hypotenuse}} = \frac{\text{ordinate}}{\text{hypotenuse}}$$

for the sine, since AB is a half chord which represents the ratio. The etymology of our word "sine" finds its beginning in the Hindu word *jyā* which Āryabhata used as an abbreviation for the word *ārdhā-jyā* meaning *half chord.*[1]

The variation of cos $\angle BOA$ may be studied by observing the behavior of the line OB in Fig. 54 as $\angle BOA$ varies from 0 to 90 degrees. The extreme values are, of course, $\cos 0° = OX = 1$ and $\cos 90° = 0$. It is also quite evident from the figure and discussion that the cosine decreases as the sine increases.

In a similar way the line XT in Fig. 55 represents the tan $\angle XOT$, $YS = \tan \angle SOY = \cot \angle XOT$, $OT = \sec \angle XOT$, and $OS = \sec \angle SOY = \csc \angle XOT$. The appropriateness of these names is immediately evident. As $\angle XOT$ decreases in size XT decreases, while the length of YS increases without limit. The evidence of this last statement can be made clear to the pupil by drawing several positions of the terminal line OTS. As the different positions of this line make $\angle XOT$ take on values nearer to 0 degrees, it will be evident that T approaches X along XT and YS becomes longer and longer as the line OTS approaches coincidence with OX. At the same time it can be demonstrated that $\sec \angle XOT = OT$ takes on the value $OX = 1$ for 0 degrees, while $\csc \angle XOT = OS$ increases without limit. The symbol for "increasing without limit" is ∞, sometimes called "infinity." Thus $\cot 0° = \infty$ and $\csc 0° = \infty$ are convenient symbols for saying that, as the angle gets nearer and nearer to 0 degrees in size, the cotangent and cosecant increase without limit. The pupil should be cautioned to remember that ∞ is merely a symbol and that it is not to be thought of as a number. It is therefore wrong to read $\cot 0° = \infty$ as "cotangent of an angle of zero degrees is equal to infinity."

By drawing the terminal line OTS in several different positions which show $\angle XOT$ approaching 90 degrees, it can be demonstrated that

$$\begin{array}{llll} \tan 90° & \infty & \sec 90° & \\ \cot 90° & 0 & \csc 90° & 1 \end{array}$$

[1] D. E. Smith, "History of Mathematics" (Boston: Ginn & Company, 1925), Vol. II, pp. 615–618.

From the foregoing considerations the values of all the functions of these special angles can be written down at once since the relative lengths of the hypotenuse and the two legs in each of the special triangles is known. This is shown in Table 6.

TABLE 6. FUNCTIONS OF SPECIAL ANGLES

	0°	30°	45°	60°	90°
sine	0	½	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cosine	1	$\sqrt{3}/2$	$1/\sqrt{2}$	½	0
tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
cotangent	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0
secant	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞
cosecant	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1

These values (except, of course, ∞, or infinity) can be readily translated into decimal equivalents if desired. The student who learns to derive the values of the functions of these special angles in this way need never be afraid of forgetting them, because he can reconstruct them at will. Moreover, being associated directly with the triangles, they will carry for him a real functional meaning and will not be mere arbitrary and unrelated numbers.

Teaching the Use of the Tables and Interpolation. Students should be given special training in the use of the tables and in interpolation. While it is not difficult to learn how to use the tables, careful explanation by the teacher and a considerable amount of practice is necessary if the students are to acquire the speed and facility required for efficient work. Consequently it is important in the beginning, and at intervals thereafter, to supplement the incidental use of the tables by special drills particularly designed to improve the students' expertness in this phase of the work. These drills should emphasize the determination of inverse functions such as arcsin, arctan, antilogarithm, etc., as well as of the functions themselves. Since most tables of natural functions and of logarithms of natural functions do not give values for the secant and cosecant, the students should be reminded that these functions may be found if needed by taking the reciprocals of the values given for the cosine and sine, respectively, or that the logarithms of these functions may be found by taking the *cologarithms* of the cosine and sine, respectively. In this connection it will be well to explain and define the meaning of the term "cologarithm" if this has not been done before. While it is possible to dispense with the use of cologa-

rithms, and some teachers prefer to do so, others find their use convenient and desirable. Moreover, whatever may be the teacher's convictions in this respect, the students will inevitably come upon the term in their reading and will be handicapped if they do not understand its meaning.

Students often fail to realize the approximate nature of most of the values given in the tables of logarithms and natural functions. It should be made clear to them that with comparatively few exceptions these values are not exact, but represent merely approximations to the exact values. Comparison of the tables which they commonly use with other published tables of greater or less accuracy will help to impress the students with this fact.

The greatest difficulty which most students encounter in using the tables is the determination of intermediate values by interpolation. Their difficulty in this connection resides not so much in the nature of the process as in the fact that too often it is not explained in its elemental meaningful simplicity but is set forth as an arbitrary, mechanical procedure. The fact is that nearly all people make intelligent use of the *principle* of linear interpolation continually in many of their everyday problems. The matter presents no material difficulty in connection with such situations as measuring, buying, and selling, because here it is related to tangible, concrete concepts and denominate quantities and the procedure is almost intuitive. It is only when it is dissociated from denominate quantities and applied to sheer numbers that it seems to lose, for many people, this intuitive character. It is desirable to retain the association and similarity between these two kinds of situations and this can best be done by explaining interpolation first in terms of the simple, concrete, familiar situations.

By way of illustration, consider the following problem:

Example. If a man's wages are 80 cents an hour, how much will he earn by working 6 hours and 45 minutes?

This can be solved, of course, by a single simple multiplication, but it may also serve to illustrate the principle of interpolation. It may be explained somewhat as follows:

The man worked $6\frac{3}{4}$ hours.

This is between 6 hours and 7 hours and is $\frac{3}{4}$ of the way from 6 hours to 7 hours.

His wages for 6 hours would be 6×80 cents or $4.80.

His wages for 7 hours would be 7×80 cents or $5.60.

By computation $\frac{3}{4}$ of 80 cents is 60 cents, and this must be added

on to the \$4.80 which represents his wages for 6 hours, so that his wages for 6 hours and 45 minutes amount to \$4.80 + \$0.60, or \$5.40.

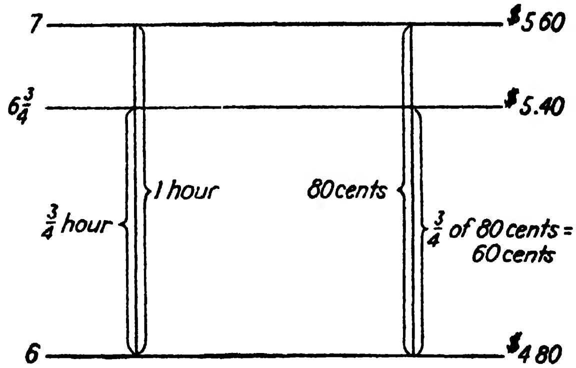


FIG 56

Example. I can buy 5 gallons of gasoline for \$1 35 or 6 gallons for \$1.62. How much can I buy for \$1.50?

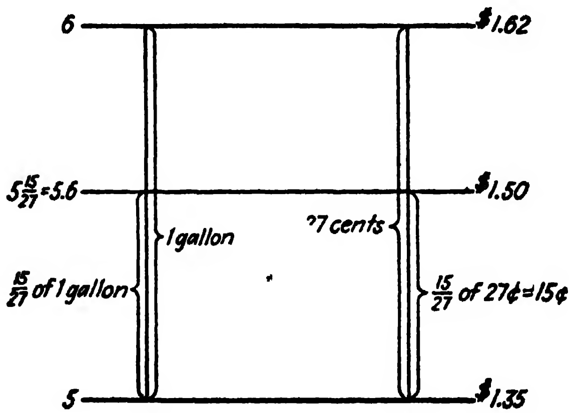


FIG. 57.

Since the cost of 5 gallons is less than \$1.50 and the cost of 6 gallons is more than \$1.50, the amount I can buy must be between 5 and 6 gallons, *i.e.*, 5 gallons and a part of another gallon.

Since \$1.50 is $^{15}/_{27}$ of the way from \$1.35 to \$1.62, then the number of gallons it will buy must be $^{15}/_{27}$ of the way from 5 gallons to 6 gallons. Thus it is easily seen that at this rate \$1.50 will buy $(5 + {^{15}/_{27}})$ gallons or approximately 5.6 gallons.

These examples bring out four important points: (1) In interpolation we always seek to find a number which lies *between two other known numbers;* (2) we may find this number by comparison with three other corresponding numbers all of which are known; (3) the differences

between the corresponding numbers in the two sets must be proportional to each other; and (4) the required difference, when found, must be properly associated by addition or subtraction with the appropriate one of the known numbers between which it lies. The use of diagrams similar to those used in the foregoing examples has been found to be extremely helpful in making these points clear to the students.

Let us now take three examples from the tables.

Example. Find the value of tan 18°21′12″.

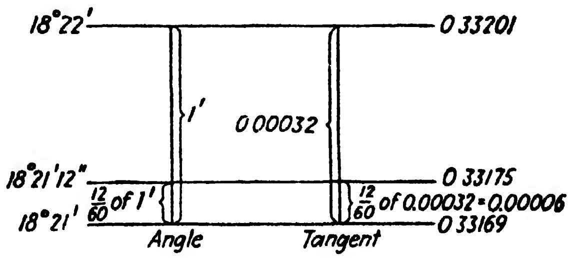


FIG. 58.

Therefore tan 18°21′12″ = .33169 + .00006 = .33175.

Example. Find log 327.38.

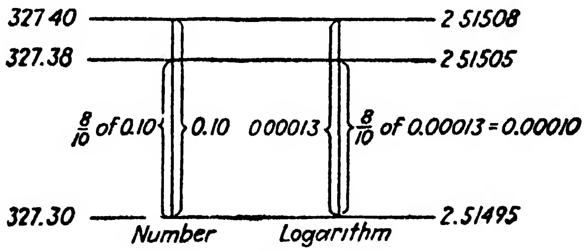


FIG. 59.

Thus, log 327.38 = 2.51495 + .00010 = 2.51505.

Example. Find the value of cos 38°14′29″.

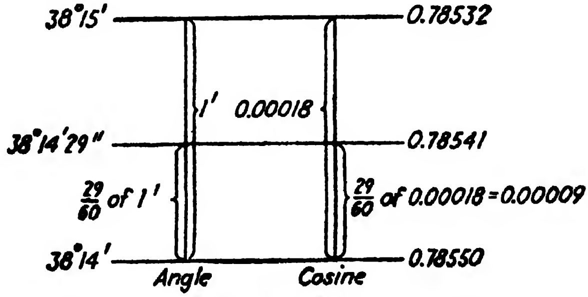


FIG. 60.

It has been pointed out in previous discussion that the cosine of an angle is decreasing as the angle increases. Consequently, to obtain the interpolated value of this function, the difference ($^{29}\!/_{60}$ of 0.00018) must be subtracted from the value of cos 38°14′ instead of being added to it.

In many texts the tables are accompanied by sets of proportional parts of differences to facilitate and speed up the process of interpolation. These are of value after the process is thoroughly understood, but they should not be used in the beginning. The understanding and use of these auxiliary tables constitutes a separate and distinct difficulty, and at the outset the student is confronted with enough of a problem in thoroughly mastering the *principle* of linear interpolation. Until he has thoroughly mastered this principle he should work through every step of it, including the computation of the proportional parts of the differences, for himself. Then, when he is ready to use the short cut provided by the sets of proportional parts printed with the tables, he will understand how they are obtained and will be able to use them with understanding.

Functions of the General Angle. There are two points of view with regard to the introduction to the study of the functions of the general angle. Some teachers and writers believe that it is best to make this the starting point of the course, and it must be admitted that there are certain arguments which tend to support this position. In particular, it provides a general situation in which the definitions of the functions of an acute angle, when given in terms of the sides of a right triangle, appear merely as special cases, and thus it avoids the necessity for extending the definitions after they are once made. On the other hand, there are numerous advantages in beginning by defining the functions of an acute angle in terms of the sides of a right triangle. In particular, these definitions are a basic part of the trigonometric background which the students bring with them from their work with numerical trigonometry in the junior high school. These definitions thus form probably the most natural point of departure in taking up the more systematic and more general treatment of trigonometric functions in the senior high school and junior college. The subsequent transition to the functions of the general angle does, of course, involve a generalization of the definition of the functions, but this need not cause any serious difficulty if it is explained carefully.

Let us assume that the students have learned to define the functions of an acute angle in terms of the sides of a right triangle as indicated earlier in this chapter.

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}}$$

Now if we draw a set of perpendicular coordinate axes such that the origin is at A with the positive X-axis passing through B, we are ready to redefine the functions of the *acute* angle A in terms of the abscissa, ordinate, and distance of C, or of any other point on the terminal side of the angle. That is, the "opposite side" in the triangle is now y, the ordinate of C; the "adjacent side" is x, the abscissa of C; and the "hypotenuse" is r, the distance of C (Fig. 61).

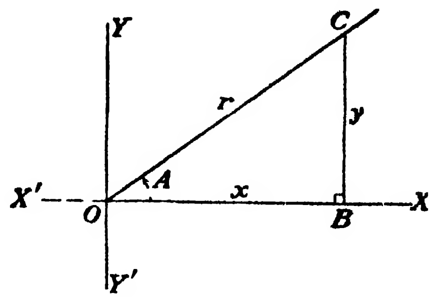


FIG. 61.

Consequently our definitions may now take the form

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{\text{ordinate}}{\text{distance}} = \frac{y}{r}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{\text{abscissa}}{\text{distance}} = \frac{x}{r}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}$$
etc.

One thing should be especially noted Thus far we have redefined the functions *only for acute angles*. What shall we do in the case of angles greater than 90 degrees or in the case of negative angles?

It should be made clear to the students at this point that our new definitions of the functions of acute angles give us a concept, a nomenclature, and a symbolism that will apply as readily to angles of any size, whether positive or negative, as it does to acute angles. This being the case, it has been generally agreed that these definitions, although developed originally for acute angles, will be accepted for the sake of uniformity and convenience (and for no other reason) as defining the functions of *any* angle It is necessary only to recall the "standard position" of the angle with reference to the coordinate axes; *i.e.*, its vertex will always be considered to be at the origin, its initial

side the positive X-axis, and the position of its terminal side determined by the description of the angle or the conditions of the problem. *The functions of any angle whatever*[1] can thus be given in terms of the coordinates of any point (except the origin) on the terminal side of the angle.

In order to use these definitions, it is necessary to have some means of transforming the functions of the general angle into terms of func-

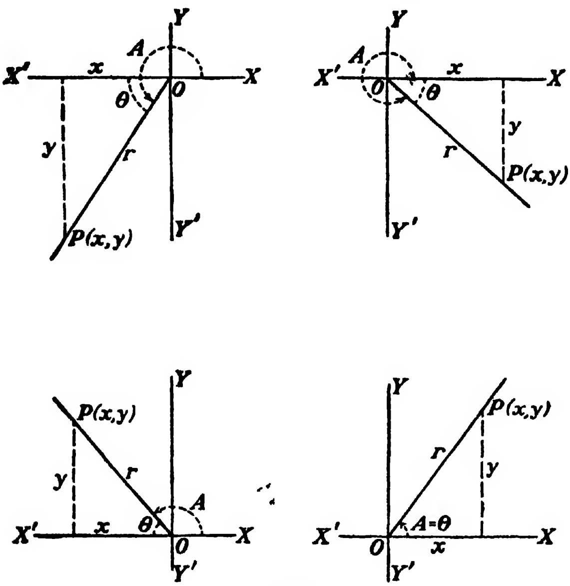


Fig. 62

tions of an angle in the first quadrant or at the boundaries of that quadrant. This is necessary because the tables of functions are generally given only for such angles. The method of making the required transformations may be effectively explained to the students somewhat as follows:

Let A be the given angle in standard position as shown (Fig. 62). If we take any point $P(x, y)$ on the terminal side of angle A and drop a perpendicular to the X-axis, there will be formed a right triangle which is called the "triangle of reference." The values of the functions of A are determined by the lengths of the sides of this triangle

[1] Of course, exception is made of those functions of 0, 90, 180 degrees, etc., which become infinite and are not so defined in any case.

since its sides are, respectively, x, y, and r, the abscissa, ordinate, and distance of P.

Now, if A is known, we may also find at once the angle θ, without regard to sign, the smallest angle which OP makes with the X-axis. This will always be the angle XOP or $X'OP$ in the triangle of reference, and its value will never exceed 90 degrees. Thus θ becomes an angle which has the limits $0° \leqq \theta \leqq 90°$, and *the values of the functions of A will be the same as the values of the functions of θ except for sign.* The functions of θ can be found in the tables, and the signs of the corresponding functions of A (whether positive or negative and regardless of size) can be determined by noting the signs of x and y (r is always positive) in the triangle of reference.

Example I. Find the value of cot 112°5′.

Let $A = 112°5'$. Then $\theta = 67°55'$. From the tables cot $67°55' = .40572$. But in the *triangle of reference* (Fig. 63) cot θ is negative. Therefore, cot 112°5′ is negative also, and its real value is $-.40572$.

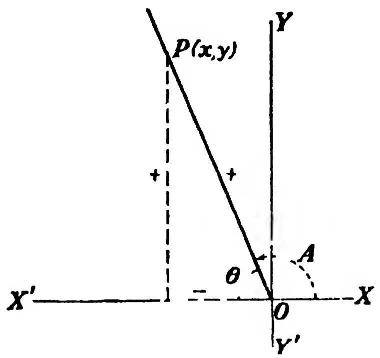


FIG. 63.

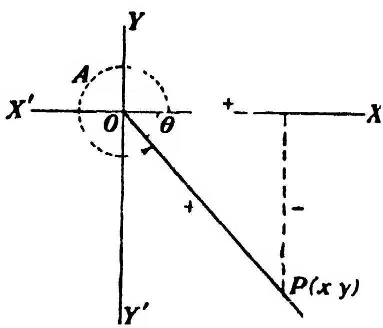


FIG. 64.

Example II. Find the value of cos 312°.

Let $A = 312°$. Thus $\theta = 48°$ (Fig. 64). From the tables cos $48° = .66913$. Therefore the value, except for sign, of cos $312° = .66913$. But in the *triangle of reference* cos θ is positive. Therefore cos 312° is positive also, and its real value is $+.66913$.

If the matter is explained and illustrated in this way, the students should have little difficulty in understanding how to find the functions of any given angle from the tables. It is well for them to work out numerous examples of the type given above and to build for themselves tables showing the signs of each of the functions when the angle is in standard position and the terminal side lies, respectively, in the following eight positions:

In the first quadrant	**Along the positive X-axis**
In the second quadrant	**Along the positive Y-axis**
In the third quadrant	**Along the negative X-axis**
In the fourth quadrant	**Along the negative Y-axis**

When the students have acquired the ability to do this for themselves they will have attained a real understanding of how the tables may be used to find the functions of any angle whatever. They should experience no further difficulty in this respect, and there will be no need for them to memorize the signs of the functions for the various positions of the terminal side of the angle as indicated in the list given above.

Variation of the Functions. Just as the concept of variation enriches the study of geometry and algebra, so it does the study of trigonometry. It is very important for the students to sense the fact that any *change* in the size of an angle is accompanied by a corresponding characteristic change in each of the functions of the angle. This fact should be repeatedly emphasized by the teacher and should be illustrated by all available means and devices. One such means is a thoughtful examination of the tables of functions. It will be immediately noted that increases in the size of the angle are accompanied by characteristic increases or decreases in each of the functions. It should be specifically pointed out that the values given in the tables merely represent particular stages in a continuous variation and that, between any two successive values given in the tables, there exists an infinitude of intermediate values. This concept of variation of the functions can be strengthened by certain mechanical devices which are available commercially, or which can be made by the teacher or students. Perhaps the most commonly used and most effective device is the construction and study of graphs of these functions.

A real understanding of these graphs makes clear several important things about the variation of the functions. In particular, the following considerations are worthy of note and comment:

1. There is a general similarity in the shape of the graph of any function and its corresponding "cofunction."[1] This may be explained by the fact that, for example, the cosine of a given angle is, in fact, the sine of some other angle (in general, the complement of the given angle), and similarly for the other functions and their corresponding cofunctions.

2. Sin θ and cos θ are always finite and continuous over the entire

[1] Here we speak of sin θ and cos θ, for example, as "*cofunctions*" *of each other.* Like reference is made to tan θ and cot θ, and to sec θ and csc θ.

range of values of θ, while for certain values of θ the other functions become infinite and have points of discontinuity. This may be explained by the fact that each of the other functions is the reciprocal of some function which may take on the value zero and, when this happens, the reciprocal obviously becomes infinite. It should also be

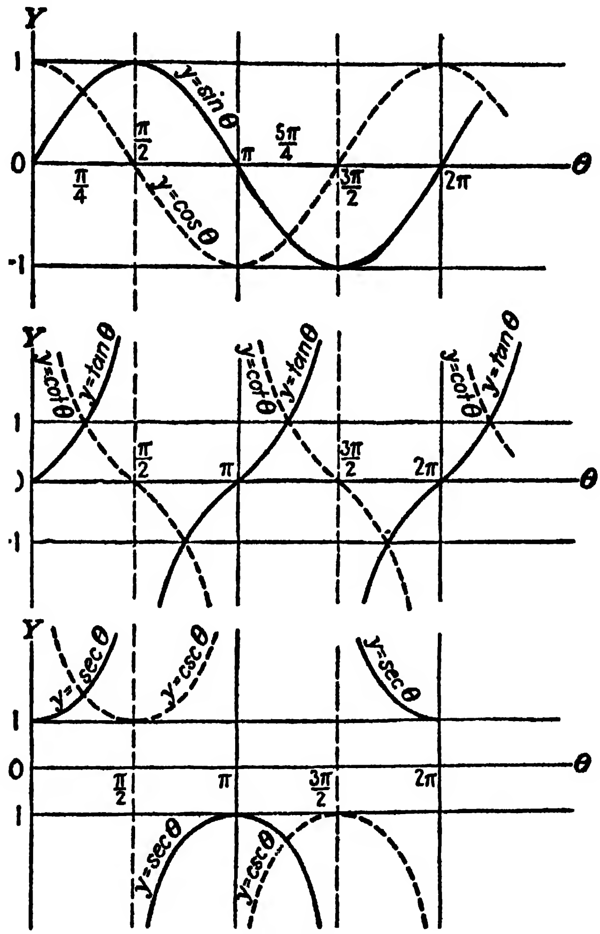


FIG 65.

noted that, at each value of the angle for which a given function becomes discontinuous, the function also changes sign.

3. The sine or cosine of an angle can never be greater than $+1$ or less than -1, while the secant and cosecant can never have values between these bounds. This is explained by the fact that the secant and cosecant of an angle are reciprocals, respectively, of the cosine and sine of the angle, and the reciprocal of any number which is not

greater than 1 or less than -1 will have a value which must be > 1 or < -1. Numerical illustrations of these facts should be given.

4. From the graphs it will be seen that, when the angle is given by $(k\pi + \pi/4)$ for all integral values of k, any function of the angle is equal to its corresponding cofunction. The students may be asked to explain this.

5. The graph of any function, say $\sin \theta$, is also the graph of the inverse function, which in this case is $\arcsin \theta$. The distinction lies

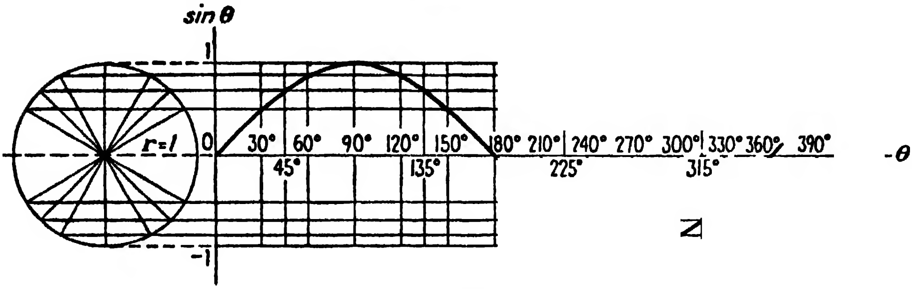


FIG. 66.

merely in the choice of the independent variable. This fact makes it easy to explain and emphasize the important concept of the multiple-valued nature of the inverse functions as contrasted with the single-valued nature of the functions themselves. The use of the unit circle in drawing the graphs of the different functions is a great aid in bringing out this contrast between the functions and their inverse functions. It also gives a vivid picture of the periodicity of the functions and the nature of their respective variations.

Practically all textbooks contain graphs of the functions for reference, and in many cases the students are required to construct their own graphs. Too often, however, the attention given to the graphs of the functions is perfunctory and not very meaningful. If the graphs are to contribute much to developing the concept and the nature of the variation of the functions, considerable time will need to be spent in the interpretation of the graphs and in the discussion of their implications.

Teaching the Functions of Two Angles. Since there are times when the students will need to use the functions of the sum or difference or two angles, of a double angle, or of a half angle and perhaps certain other formulas involving two angles, it is necessary that these formulas be developed in terms of the original angles and listed for reference. This is done in practically all textbooks. It is not necessary that students memorize all of these formulas. It is desirable,

however, that they see and understand the ways in which the formulas are developed and that they memorize a few which are fundamental and which will be needed frequently. Among these may be listed the following:

$$\begin{array}{lll} \sin (A \pm B) & \cos (A \pm B) & \tan (A \pm B) \\ \sin 2A & \cos 2A & \tan 2A \end{array}$$

The development of a large array of these formulas is an excellent exercise in the application of algebraic technique to trigonometric functions and can be made extremely interesting to students if rightly presented. After the formula

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

has been established, all the rest of these formulas may be derived from this one through the application of fundamental algebraic and trigonometric techniques. For example:

$$\begin{aligned} 1.\ \sin (A - B) &= \sin [A + (-B)] \\ &= \sin A \cos (-B) + \cos A \sin (-B) \\ &= \sin A \cos B - \cos A \sin B \\ 2.\ \cos (A + B) &= \sin [90° - (A + B)] \\ &= \sin [(90° - A) - B] \\ &= \sin (90° - A) \cos B - \cos (90° - A) \sin B \\ &= \cos A \cos B - \sin A \sin B \end{aligned}$$

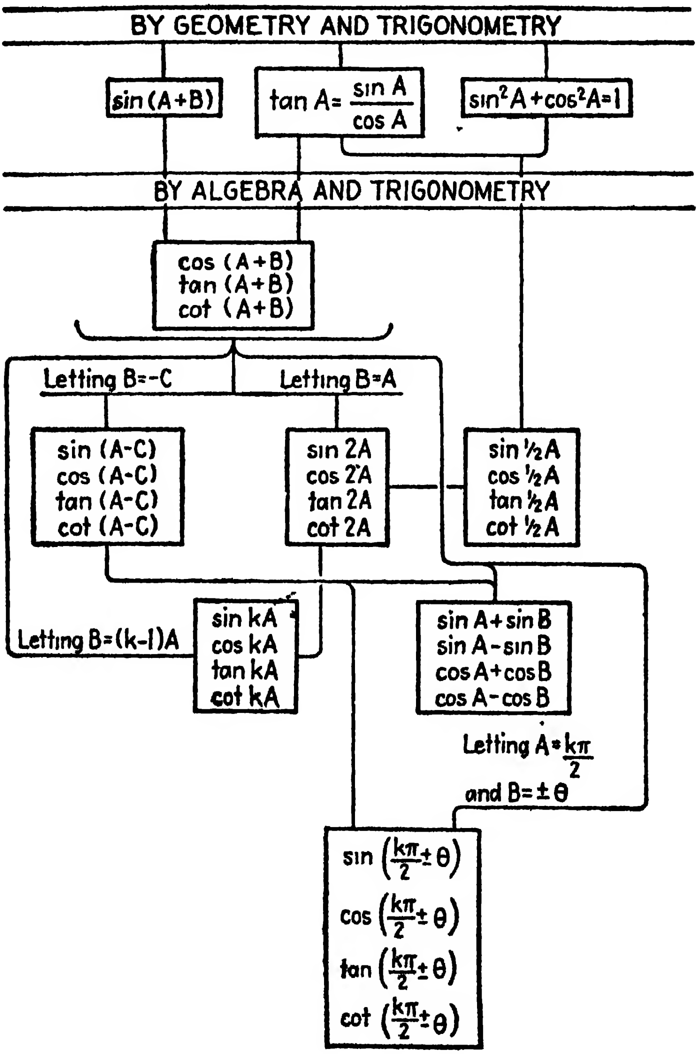
It is surprising that in the teaching of these formulas this fact is almost never emphasized either in the textbooks or by the teachers. One would naturally expect students to be dismayed at the prospect of memorizing arbitrarily 20 or more special formulas, but one would expect them to be fascinated at seeing these all stem from a single one. Yet we persist in teaching them as cases unto themselves without stressing their interrelations, and thus we fail to take advantage of the most powerful of motives in our teaching, and we likewise fail to give our students that sense of organic relationship which they should feel with regard to this whole group of formulas.

As has been said, the formula for $\sin (A + B)$ must be developed independently from geometric considerations. Table 7 shows how all the others grow out of this one.

The Teaching of Logarithms. It should be made clear to students that logarithms are not an integral part of trigonometry but merely provide a means for reducing the laborious computation incident to the solution of many trigonometric problems. Many students in

trigonometry will have had no previous experience with logarithms. Therefore it is usually necessary to start at the very beginning and to teach the meaning of logarithms first. The technique of applying them to the solution of problems can be intelligently used only after the meaning is clear.

TABLE 7. REDUCTION FORMULA



It is best to start out by explaining that logarithms are merely exponents, that a table of exponents can be useful in making certain computations, and that we merely apply the laws of exponents when

we use logarithms for finding powers and roots of numbers and for multiplying or dividing numbers expressed as powers of a common base. A review of these fundamental properties, laws, and uses of exponents can be made by taking some convenient small positive integer as a base and building a partial table of exponents of numbers with respect to that base. For example:

Base 2	Base 8
$2^0 = 1$	$(8)^{0/3} =$
$2^1 = 2$	$(8)^{1/3} =$
$2^2 = 4$	$(8)^{2/3} =$
$2^3 = 8$	$(8)^{3/3} = 1$
$2^4 = 16$	$(8)^{4/3} = 1$
$2^5 = 32$	$(8)^{5/3} = 3$
$2^6 = 64$	$(8)^{6/3} = 6$
etc.	$(8)^{7/3} = 12$
	$(8)^{8/3} = 25$
	etc.

By using the first of these tables (extended if desired), simple operations in finding powers and roots, products, and quotients can be performed. Thus,

$$\sqrt[3]{64} = \sqrt[3]{2^6} = (2)^{6/3} = 2^2 = 4$$
$$4 \cdot 4 = 2^2 \cdot 2^2 = 2^{(2+2)} = 2^4 = 16$$
$$64 \div 4 = 2^6 \div 2^2 = 2^{(6-2)} = 2^4 = 16$$

It will be obvious to the student, however, that these tables or any other tables similarly constructed would be very limited in their application. For example, the first table above contains no logarithms for 3, 5, 6, 7, 9, 10, or any other number which is not an integral power of 2. The logarithms of such numbers, with reference to the base 2, would necessarily be fractional since they lie between numbers which are integral powers of 2. A similar observation should be made, of course, with reference to a table similarly constructed with any positive number as a base. However, the students should also be told that, although they do not know how to determine these fractional exponents, there are methods by which this can be done, and really serviceable tables compiled. They should also be reminded that, since our number system is a decimal system, the number 10 would seem to be a more convenient base than 2 or 8 or any other number, because the integral powers of 10 give our familiar units of enumeration, *viz.*, 1, 10, 100, 1,000, etc.

By this time the stage is set for introducing the students to the system of common logarithms in which the base is 10. The following

step-by-step procedure has been found effective in producing understanding of this system and skill and facility in its use. The order of the steps is important since each new idea or principle developed rests upon those preceding it and, in turn, provides an additional idea or principle as a basis for those yet to come. The key sentence or idea in each step is italicized. The teacher may enlarge upon these italicized statements and give such explanations, illustrations, and practice as may be desired. Some suggestions are given.

1. *Any number in the decimal system can be expressed as an integral power of* 10 *or as the product of two factors, one of which is between* 1 *and* 10, *the other being an integral power of* 10.

Give examples. For instance,

$$32.78 = 3.278 \times (10)^1 \qquad 35{,}200 = 3.52 \times (10)^4$$
$$0.346 = 3.46 \times (10)^{-1} \qquad 0.00682 = 6.82 \times (10)^{-3}, \text{ etc.}$$

Give considerable practice in writing numbers in this way. Such consideration would afford further opportunities for calling attention to the nature and importance of scientific notation (see page 283). If the students have difficulty in determining the integral power of 10 to be used, the following rule will be helpful: place the pencil point after the nonzero digit on the extreme left of the given number and count to the position of the decimal point. The number of digits (or places) thus counted gives the integral power of 10 to be used. Counting to the right in this way indicates a positive power of 10, while counting to the left indicates a negative power of 10. It may be noted in passing that this integral power of 10 is the characteristic of the logarithm of the number, but it is better not to use the terms "logarithm" or "characteristic" until a little later.

2. *If we could express all numbers between* 1 *and* 10 *as powers of* 10, *then it is evident that we could express all numbers in the decimal system as powers of* 10. Point out that this follows from the fact that every number in the decimal system can be expressed as an integral power of 10 or as the product of such a power of 10 by a number between 1 and 10. Refer to the illustrations under step 1 above.

3. *Any number between* 1 *and* 10 *can be expressed either exactly or approximately as a power of* 10. Do not try to prove this but make it appear plausible. Remind the students that since $1 = (10)^0$ and $10 = (10)^1$, then surely any number between 1 and 10 must be expressible (at least approximately) in the form $(10)^x$, where $0 < x < 1$. For

example, if $x = \frac{1}{3}$, we have $(10)^{\frac{1}{3}} = 2.154$ (approximately), or 2.154 is approximately ten to the one-third power. Similarly $(10)^{\frac{1}{2}} = 3.162$ (approximately); $(10)^{\frac{2}{3}} = 4.641$ (approximately); etc.

4. *Mathematicians have given us a table of numbers between* 1 *and* 10 *expressed as powers of* 10. Make it clear that the numbers in the left margin of the table of logarithms should be read as if there were a decimal point after the first digit at the left in the number. This is, in fact, the number for which the logarithm is given in the table, since only mantissas are given, and these, themselves, are logarithms only of numbers between 1 and 10. This number will be the first of the two factors referred to in step 1. It may also be explained to the students that the omission of the decimal point in the numbers printed in the tables makes for convenience and for the conservation of space. When the students have learned to read and interpret the numbers in the left margin of the table of logarithms in this way, have them refer to their tables of mantissas and read from these the powers of 10 representing various numbers between 1 and 10. For example, $5 = (10)^{.69897}$; $3.27 = (10)^{.51455}$; $7.84 = (10)^{.89432}$; etc. For the present have the students think of these as exponents rather than as logarithms. Point out that the decimal point precedes the first figure in the number taken from the table, whether it is printed there or not. Give practice in reading and writing down these exponents and in interpreting them.

5. *We can make certain computations by use of this table alone so long, and only so long, as the numbers we use are between* 1 *and* 10 *and the answers are also between* 1 *and* 10. Give practice, using examples in multiplication and division and in finding powers and roots, but be sure that at this stage no number used or answer sought is less than 1 or greater than 10.

6. *We may now use our tables for any positive numbers.* Give practice in multiplication, powers, roots, and appropriate division, but for the present avoid division that would lead to negative exponents. This is a problem in itself. In connection with the use of the tables for the inverse cases, teach the students to express the numbers as the product of two factors. For example, have them express $(10)^{3.82367}$ as $(10)^{.82367} \times (10)^{3}$.

7. *We are now ready to begin thinking, speaking, writing, and working in terms of "logarithms" instead of "exponents."* Some practice, but not a great deal, will be needed now in reviewing and associating the logarithmic form with the exponential form. Examples:

EXPONENTIAL FORM	LOGARITHMIC FORM	
$8 = 2^3$	$\log_2 8 = 3$	(the logarithm is the exponent)
$100 = 10^2$	$\log_{10} 100 = 2$	(the logarithm is the exponent)
$a = b^c$	$\log_b a = c$	(the logarithm is the exponent)
	etc.	

Explain to the students that in the common system of logarithms the base (10) is understood and is generally not written.

Now give considerable practice in the use of the logarithmic form and notation in finding products, roots, powers, and quotients. Negative characteristics should still be avoided.

Example. Find the product of 782 and 3 96.

$$\begin{aligned} \log 782 &= \\ \log 3.96 &= \\ \hline \log \text{product} &= \\ \text{Product} &= \end{aligned}$$

At this stage of development it is probably advisable to emphasize that previously the practice work was for understanding, but now it is mainly for efficiency.

8. *Special difficulties with logarithms with negative characteristics.* Go back to the exponential form for a little while.

Example. Find log .0342.

$$.0342 = 3.42 \times 10^{-2} = (10)^{.53403} \times (10)^{-2} = (10)^{-2+.53403}$$

This, however, cannot be written as $(10)^{-2.53403}$ because the .53403 is positive. It is sometimes written as $(10)^{\bar{2}.53403}$. A more convenient way of writing the same thing is $(10)^{8.53403-10}$. In logarithmic form this would be given as log .0342 = 8.53403 − 10.

Considerable special practice will be needed on this, and it should be very carefully supervised until the students have acquired a substantial understanding of the nature of negative logarithms and reasonable proficiency in handling the special notation which it is necessary to use.

Having learned how to use the table of natural functions and the table of common logarithms, the student should have no difficulty in understanding the meaning or mastering the use of the tables of logarithms of the trigonometric functions. These tables are not, in fact, indispensable but are provided merely to speed up the work. The question of whether or not to use cologarithms is debatable. Some teachers and writers feel that cologarithms are nonessential and superfluous, while others prefer to have them used. This matter is prob-

ably of little moment and may safely be left to the judgment of the individual teacher.

An explanation of the slide rule, showing how its construction and use is based upon the principles of logarithms, will add interest and value to the study of logarithms.

The Solution of Triangles. The trigonometric solution of the right triangle has been discussed in the first part of this chapter. The methods are so simple, obvious, and direct that they offer no difficulty and need not be discussed further here. The solution of the oblique or general triangle, however, is not so simple. It requires the development of numerous special formulas and a discriminating analysis of the given data in each particular problem to determine the method and the formulas to be used.

There are four different combinations of independent parts, any one of which may determine uniquely (with one exception which will be mentioned later) a triangle. These are often referred to as "the four cases" of the general[1] triangle. The four combinations of parts are as follows:

Case 1. Given two angles and any side
Case 2. Given two sides and an angle opposite one of them
Case 3. Given two sides and the included angle
Case 4. Given the three sides

To handle the solution of these cases of the general triangle, various special formulas are developed, the most fundamental of which are the law of sines, the law of cosines, and the law of tangents. In addition to these there are numerous more specialized formulas connected with the triangle, including the half-angle formulas in terms of the sides, and Heron's formula for the area of the triangle.

The mastery of these formulas and their use in solving the various cases of the general triangle cause a good deal of difficulty to students. The difficulties which they most commonly experience may be enumerated under five main types as follows:

1. Difficulty in following and understanding the derivation of the formulas
2. Difficulty in remembering the various formulas and keeping them in mind without confusing them
3. Difficulty in knowing what formulas to use in particular cases

[1] The term "general triangle" is used here instead of "oblique triangle" since the general formulas developed for these cases apply to right triangles as well as to oblique triangles.

4. Difficulty due to lack of systematic preplanning and layout of written work
5. Difficulty in checking the solution

Many teachers fail to appreciate either the seriousness of these difficulties or the extent to which failure to master them may impair the effectiveness of the students' work. Yet the teacher has a serious obligation to see that the students are adequately equipped with respect to these points.

The development of the fundamental trigonometric laws is not inherently difficult or complicated. It can be understood without great difficulty by most students if properly presented. The teacher, however, should point the way by laying out the main steps in the development in quick bold outline just as one lays out the plan of proof for a proposition in geometry before proceeding to write down the complete synthetic proof in finished form. Some geometry teachers and textbook writers have recognized the importance of setting forth the main plan of a proof before proceeding to details. For the most part, trigonometry textbooks have not yet arrived at this point, and so the responsibility must be accepted by the teacher. For example, most students find it difficult to take the bare facts set forth in the proof of the law of cosines and to extract from them much understanding of a plan whereby the proof could be reconstructed without being literally memorized. On the other hand, a rapid "chalk-talk" explanation of the *plan* of development by the teacher will leave the student with a basis of understanding which he can fill in with the necessary details and which he can reconstruct at will. The same may be said for the development of the law of tangents and for the other formulas involved in the solution of the general triangle. In some cases it is necessary to draw upon previously developed formulas. For example, in developing the law of tangents, it is convenient to use the relations

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$$
$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$$

In such cases the teacher should specifically call attention to these substitutions in discussing the plan of development.

There are a few fundamental formulas which are used so much in the solution of oblique triangles that the student will find it advisable to memorize them. In particular, the law of sines, the law of cosines, and the law of tangents are indispensable, and the student should become as familiar with them as with the Pythagorean theorem, or the fact that $a^2 - b^2 = (a - b)(a + b)$. Some teachers prefer to have

their students memorize other formulas such as the half-angle formulas (in terms of the sides) and the area formulas. Since courses differ in emphasis no rigid criterion can be laid down. In general it may be said that students should be required to memorize only such formulas as are fundamental and which, in the judgment of the teacher, will be used sufficiently to warrant memorization; however, those formulas which *are* to be memorized should be memorized to the point of perfection, automatization, and permanency.

The teacher should give the students criteria for determining which formulas to use in particular cases. One criterion is as follows: the formula to be used must be one which expresses an unknown part of the triangle completely in terms of the given parts, or from which an unknown part may be found in terms of given parts (as in the case of the law of tangents). By adopting a standard system of notation, listing the fundamental formulas, and setting down the given parts of the triangle in a particular problem, the student may readily determine by inspection which of the formulas should be used. If he is required to do this consistently, he will soon learn that the law of sines is the appropriate formula to use, given (1) two angles and any side or (2) two sides and an angle opposite one of them;[1] and that the law of cosines will enable him to solve triangles in which the given parts are (1) two sides and the included angle, or (2) the three sides. The teacher should point out that, in place of the law of cosines, the alternative formulas for the law of tangents and for the tangents of the half angles in terms of the sides may be used where the given parts are, respectively, two sides and the included angle or the three sides, and that these formulas are more amenable to logarithmic treatment than is the law of cosines. The association of the appropriate formulas with the different cases would be strengthened if each formula were developed when the case requiring it is first taken up and were immediately applied. In many textbooks, however, the formulas are developed as in an intact body of theory, and the applications are left until later.

In teaching the solution of the general triangle, the teacher should show the students how to make a layout for the work and should give specific training in doing this. Students generally are inclined to start computing as soon as they get hold of two numbers with which they can work. However, since most problems in solving oblique triangles are somewhat long and complicated, this is uneconomical and unwise for several reasons. There is great advantage in thinking the prob-

[1] The ambiguous case (given two sides and an angle opposite one of them) should be discussed at length and its geometric and trigonometric possibilities pointed out.

lem through completely and planning the entire solution in detail, but as a whole, before doing anything else. This will ensure an understanding analysis of the problem as a whole. It will prevent the student from becoming confused in a heterogeneous mixture of analysis and computation. It will provide a specific step-by-step guide for all necessary computation and use of the tables. It is economical of time and labor. Finally, it is conducive to orderliness which, in turn, is always conducive to effective work.

The following example[1] is illustrative of what may be done in this respect. The layouts will obviously vary in detail according to the conditions and the requirements of the problem, but the idea is perfectly general and can be adapted as needed.

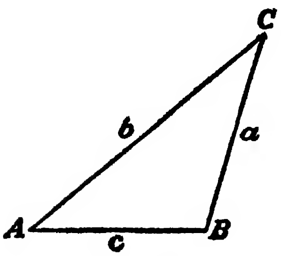


FIG. 67.

Given: $a = 81.6$.
$b = 121.5$.
$C = 32°18'$.

Find: A, B, and c.

SOLUTION

Formulas:

$$\left.\begin{aligned} B + A &= 180° - C = \quad ° \\ \tan \tfrac{1}{2}(B - A) &= \frac{b - a}{b + a} \tan \tfrac{1}{2}(B + A) \end{aligned}\right\} \text{to find } A \text{ and } B$$

$$c = \frac{a \sin C}{\sin A} \quad \ldots\ldots\ldots\ldots\ldots \quad \text{to find } c$$

$$c = \frac{b \sin C}{\sin B} \quad \ldots\ldots\ldots \quad \text{to check the work}$$

$a = 81.6$		log =	
$b = 121.5$			log =
$b - a = 39.9$	log =		
$b + a = 203.1$	colog =		
$C = 32°18'$		log sin =	log sin =
$\tfrac{1}{2}(B + A) = 73°51'$	log tan =		
$\tfrac{1}{2}(B - A) =$	log tan =		
$B =$			colog sin =
$A =$		colog sin =	
$c =$		log =	log =

[1] Adapted from Davis and Chambers, "Brief Course in Plane and Spherical Trigonometry" (New York: American Book Company, 1933), pp. 60*ff.*
See also C. A. Ewing, "Plane Trigonometry" (New York: McGraw-Hill Book Company, Inc., 1933), pp. 49–50, 70*ff.*

This outline provides a complete guide for the solution and checking of the problem.

The check used in the above outline is one of several special checks adapted to particular cases. Either of the formulas

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}, \quad \frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C},$$

called Mollweide's equations, may be used as a general check of the solution of any triangle since they contain all the sides and all the angles of the triangle, are well adapted to logarithmic computation, and are not used, in any of the four cases, in the solution of the triangle.

Table 8 gives a very concise summary of the four cases of the general triangle.

TABLE 8. FOUR CASES OF THE GENERAL TRIANGLES*

	Case I	Case II	Case III	Case IV
Given	One side and two angles	Two sides and the angle opposite one of them	Two sides and the included angle	Three sides
Solution (by logarithms)	Law of Sines	Law of Sines	Law of Tangents	Tangent half angle formula
Check	$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}$			$A + B + C = 180°$
Solution (by natural functions)	Law of Sines	Law of Sines	Law of Cosines†	Law of Cosines†
Check	$a = b \cos C + c \cos B$		Law of Sines or $A + B + C = 180°$	$A + B + C = 180°$
Area	$\frac{1}{2} a^2 \frac{\sin B \sin C}{\sin A}$	$\frac{1}{2} ab \sin C$		rs or $\sqrt{s(s-a)(s-b)(s-c)}$

* Adapted from Garabedian and Winston, 'Plane Trigonometry' (New York, McGraw-Hill Book Company, Inc, 1929), p 290.

† The Law of Cosines is usually preferable if only the third side in case III, or one angle in case IV, is to be found

Teaching Radian Measure. The teaching of radian measure should not be difficult provided that the students have a good understanding

of the functions of the general angle and provided that two or three fundamental concepts and conventions are made clear at the outset. In order to understand radian measure clearly the students must first of all have a thorough understanding of the definition and meaning of a radian. The geometrical explanation and illustration should be given repeatedly until the students can readily form a mental picture similar to the one shown in Fig. 68. This will provide them with a concrete basis for thinking in terms of radians and for understanding the definition. It will provide a means for recalling instantly why the angle π is the same as 180 degrees; why $\pi/2$ is the same as 90 degrees; why $\pi/3$ is the same as 60 degrees; why 2π is the same as 360 degrees; etc. With this sort of picture in mind for reference the student will have no need to memorize the relations between degrees and radian measure, because he can easily figure out these relationships for himself. He is thus equipped, not only with a knowledge of the facts and relationships which he will need, but also with an understanding of how they are determined.

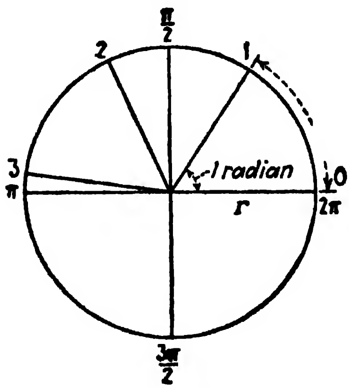


Fig. 68.

After the students have gained an understanding of the meaning of radian measure, it is well for them to have some special practice in determining the degree equivalents of angles expressed in radians, and the radian equivalents of angles expressed in degrees. In this connection it may be desirable to have the degree value of one radian memorized. For the special angles 0, $\pi/2$, $\pi/3$, π, 2π, 45 degrees, 120 degrees, etc., the students should soon become able to give the corresponding equivalents at sight. Similar practice involving the inverse functions should be given, the principal values of the angles being given both in radians and degrees. The teacher should explain that, in expressing angles in terms of radian measure, it is customary to omit the word "radians." The student needs to be told this so that, when he encounters in his reading such expressions as sin π, tan 2, cos $3\pi/4$, etc., he will understand that the angles are given in radians rather than in degrees. It may be pointed out that in higher mathematics radian measure is used almost exclusively.

Teaching Trigonometric Equations and Identities. The proof of trigonometric identities and the solution of trigonometric equations

are often considered to be the most difficult parts of trigonometry. Confusion often exists in the minds of students as to the distinction between identities and equations of condition. Therefore this distinction should be cleared up at the outset.

A trigonometric equation is an equation involving one or more trigonometric functions of the unknown (angle) and is true only for particular values of this unknown. In contrast to such conditional equations there is another class of trigonometric equalities which are true for *all* values of the unknown. These are known as trigonometric identities. From these considerations it is evident that, in dealing with trigonometric equations, the job is to *solve* the equation; *i.e.*, to find the value, or values, of the unknown which will satisfy the equation. On the other hand, in dealing with identities, the problem is to prove that they actually are identities; to show that the two members can be reduced to the same expression or expressions which are recognized as being identical.

Trigonometric equations really are equations, and therefore are subject to treatment under the algebraic laws of the equation besides being subject to the algebraic transformation of either member separately or to the trigonometric transformation of the trigonometric elements involved. For example, let it be required to solve:

$$\sin 2x = \tan x.$$

The problem is to find the angle x for which this relation exists. A solution is as follows:

$\sin 2x = 2 \sin x \cos x$	(trigonometric transformation)
$\tan x = \dfrac{\sin x}{\cos x}$	(trigonometric transformation)
$\therefore\ 2 \sin x \cos x = \dfrac{\sin x}{\cos x}$	(substitution)
$2 \sin x \cos^2 x = \sin x$	(both members multiplied by $\cos x$)
$2 \sin x \cos^2 x - \sin x = 0$	(transposition)
$\sin x(2 \cos^2 x - 1) = 0$	(left member factored)

This can be true only (1) if $\sin x = 0$ or (2) if $2 \cos^2 x - 1 = 0$.

$\therefore$ the equation is true for the values

(1) $x = 0°$ or $180°$

(2) $x = 45°, 135°, 225°$, or $315°$

and for no other values of x where $x \leqq 360°$.

Note that the algebraic transformations which were employed are

all entirely legitimate since we started with an admitted equation and sought merely the solution of this equation.

In practically all textbooks the trigonometric identities proposed for proof are stated in the form of equations. This tends to give students a wrong feeling about the job of proving these identities. Since they are stated in the form of equations, the tendency is to feel that they may be treated legitimately under the algebraic laws of the equation. This, of course, is not legitimate. Consider, for example, the following illustration which is typical of the form in which these identities are proposed in the textbooks:

Prove:

$$\frac{2 - \cos x}{\cos x} = 2 \sec x - 1 \tag{A}$$

Suppose that this were regarded as an equation and treated as such in the following manner: multiplying through by $\cos x$ we get

$$2 - \cos x = 2 \sec x \cos x - \cos x$$

or

$$2 - \cos x = 2(1) - \cos x \tag{B}$$

This evidently gives an identity, but it is not a proof that $\frac{2 - \cos x}{\cos x}$ and $2 \sec x - 1$ are identical. Rather, it is a proof that, if equation (A) is an identity, then equation (B) is an identity. In effect, what has been done here is to assume the equality of the members of the proposed identity, to treat them algebraically under the laws of the equation, and to conclude that, since an identity is thus reached, the original pair of expressions must also have been identical.

The fallacy involved in this method of reasoning may be shown even more clearly by taking a case which is evidently absurd on the face of it but which illustrates precisely the point in question. Let it be required to test to see whether or not $\sin x = -\sin x$ is an identity. By treating this as an equation and squaring both members, we do get an identity, *viz.*, $\sin^2 x = \sin^2 x$. The foregoing line of reasoning would lead us to conclude that, since this is true, therefore the original pair of expressions ($\sin x$ and $-\sin x$) must also have been identical. Obviously, however, we should not be justified in such a conclusion because the equation $\sin x = -\sin x$ is not true for all values of x. It holds only when $x = k\pi$ (k being any positive or negative integer, or zero). The above technique establishes a valid proof of an identity only when it is reversible. Only in such a case would it be proved

that one could start with the known identity and derive the unknown one. It is evident from the above discussion that this technique provides a method which may be used to derive new identities from known identities. One must be careful, however, to use only legitimate operations at each stage of the development.

In proving two trigonometric expressions identical we may legitimately work with only one of them at a time. *Either* may be changed through legitimate operations of trigonometric transformation, factoring, multiplication of factors, collection of terms, or substitution, into the form of the other, or *both* may be changed separately into some third form. When an identity is reached by one of these methods, and only then, the identity of the original expressions is really proved. It should be evident that one could use the same general techniques to establish the difference of two given expressions as *identically* equal to zero and thus establish the identity of the expressions themselves. Although the textbooks are not clear at times on this matter, the students should be strongly impressed with this principle and with the important distinction between proving a trigonometric identity and solving a trigonometric equation.

Probably the greatest difficulties which students have in working with trigonometric equations and identities are (1) the lack of systematic methods of attack and (2) insufficient familiarity with the fundamental trigonometric identities. The following suggestions will be helpful to them in planning and carrying through their work:

1. If functions of two angles are involved, transform these so that they will all be expressed in terms of one angle. For example, in proving that $\sin 2\theta = 2 \cos \theta/\csc \theta$, one should start by expressing $\sin 2\theta$ as $2 \sin \theta \cos \theta$, so that all functions involved would be functions of the angle θ.

2. In solving equations it is generally desirable and often necessary to express all functions involved in terms of a single function by means of the fundamental identities. For example, in solving the equation $\cos x + \sec x = 5/2$ the first step would be to express $\sec x$ as $1/\cos x$. In many cases this is also helpful in proving identities.

3. Having reduced all functions involved in an equation to terms of a single function, transpose all terms involving this function to one side of the equation.

4. Having solved an equation for the value of some one function of the unknown (angle), it may be possible to determine the angle by inspection and consideration of the values of the functions of special angles. Otherwise it will be necessary to refer to the table of natural functions.

5. In proving an identity, try to transform one of the members into an expression identical with the other.

6. If it appears that this cannot be done, then try to transform both members separately into some third identical expression.

7. If other methods fail, try to obtain an identity by transforming all functions involved into terms of the sine and cosine.

Success in solving trigonometric equations or in proving identities requires close familiarity with the relations among the functions. The eight fundamental identities most commonly needed are:

(1) $\sin^2 x + \cos^2 x = 1$	(5) $\cot x = \dfrac{\cos x}{\sin x}$
(2) $\sec^2 x - \tan^2 x = 1$	(6) $\cot x = \dfrac{1}{\tan x}$
(3) $\csc^2 x - \cot^2 x = 1$	(7) $\sec x = \dfrac{1}{\cos x}$
(4) $\tan x$	(8) $\csc x = \dfrac{1}{\sin x}$

These should be thoroughly memorized. The student should be familiar already with formulas (6), (7), and (8), since they are merely the reciprocal relations pointed out in connection with the definition of the functions, and his attention should be called to the fact that (2) and (3) follow immediately from (1). In addition to these, the student should also be familiar with the functions of the double angles.

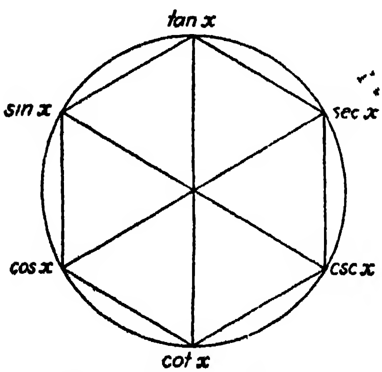


Fig. 69.—The function hexagon.

There are other relationships which are sometimes needed but which are so numerous and so much alike that it is difficult to memorize them all. The mnemonic device of Fig 69, which we shall call "the function hexagon," provides an easy means for recalling all of these fundamental identities except the first three listed above and the functions of double and half angles. The arrangement should be made exactly as shown. This is easy to remember since the sine, tangent, and secant appear in this order from left to right and each of these is directly above its corresponding cofunction.

1. The two functions at the ends of any diagonal are reciprocals of each other.

2. Any function is the product of the two functions between which it lies. Thus $\sin x = \tan x \cos x$; $\cot x = \cos x \csc x$; etc.

3. Any function may be expressed as a fraction in which the numerator is either of the adjacent functions and the denominator is the one on beyond that. Thus

$$\tan x = \frac{\sec x}{\csc x}; \qquad \cos x = \frac{\cot x}{\csc x}; \text{ etc.}$$

4. The product of any three alternate functions is 1. Thus

$$\sin x \cot x \sec x = 1 \qquad \text{and} \qquad \cos x \tan x \csc x = 1.$$

This mechanical device is so simple, economical, and powerful, giving (as it does) 26 separate identities for reference, that students will find it extremely helpful in making trigonometric transformations in their work with trigonometric identities and equations.

Some Applications of Trigonometry. The interest of the students in any course in mathematics will be intensified if genuine applications of the subject are pointed out, and trigonometry has many applications. Outside of its use in the development and application of further mathematical theory, it doubtless finds its most commonplace practical application in the work of the surveyor. This is so much the case that a large share of the verbal problems in trigonometry textbooks are problems which involve the determination of distances, angles, or areas in relation to land measurements; in other words, problems typical of those which the surveyor encounters in his work.

Nevertheless, there are many applications of trigonometry to physics, engineering, astronomy, geology, navigation, and aviation, which, though less well known, are none the less interesting. Some are simple, direct, and concrete. Others, such as some of the applications to mechanics and electrical theory, are implicit and abstract, but they are still genuine and important practical applications of trigonometry. An analysis, made some years ago, of 20 representative textbooks (5 in higher mathematics, 5 in the theory of electricity, 5 in surveying, and 5 in mechanics) for colleges of engineering gives what is probably a fair picture of the use of trigonometric functions and formulas in these fields. Table 9 (page 500) is adapted from the report of the analysis and shows the frequency of occurrence, the percentage of occurrence, and the number of trigonometric formulas found in the textbooks in each field. Textbooks in these fields will yield numerous examples of the application of trigonometry to problems in engineering practice and allied fields.

The following are a few examples, taken at random, of physical

formulas which involve trigonometric functions. They are given here merely for the purpose of giving some indication of the general usefulness of trigonometric functions. For proper use of these formulas it is necessary, first, that they be imbedded in the context of careful definition of units.

TABLE 9. OCCURRENCE OF TRIGONOMETRIC FORMULAS IN TWENTY TEXTBOOKS FOR COLLEGE ENGINEERING COURSES*

	Occurrence	Per cent	Number of formulas
Five higher mathematics texts.	314	36.7	41
Five texts in mechanics.	299	35.0	23
Five texts in surveying.	125	14.6	19
Five texts in electricity.	118	13.7	13
Total.	856	100.0	

* William Herbert Edwards, Trigonometric Formulae Encountered in a College Engineering Course, *School Science and Mathematics*, **28** (1928), 239–243.

For a coil rotating uniformly in a magnetic field, the instantaneous induced voltage (E_i) is given by the formula $E_i = E_m \sin \theta$, where E_m is the maximum voltage and θ is the phase angle.

The current i (amperes) flowing through a tangent galvanometer having n turns, of radius r, in the earth's field H, produces an angular deflection θ. The formula for the determination of the amount of current is

$$i = \frac{Hr}{2\pi n} \tan \theta$$

The index of refraction of light, n, is given by the formula

$$n = \frac{\sin i}{\sin r},$$

where i is the angle of incidence and r is the angle of refraction.

For light passing at the angle of minimum deviation, D, through a prism whose angle is A, the index of refraction, n, is given by the formula

$$n = \frac{\sin \frac{1}{2}(A + D)}{\sin \frac{1}{2}A}$$

One of the basic problems of automotive engineering is the determination of the force P needed to balance a weight W on a plane

inclined to the horizontal at a given angle α. This relationship is expressed in the formula $P = W \sin \alpha$.[1]

The motion of projectiles is another general problem involving trigonometric formulas. For projectiles having an initial velocity v at an angle θ with the horizontal the following formulas hold:
Maximum height:

$$h = \frac{v^2 \sin^2 \theta}{2g}$$

Time to maximum height:

$$t = \frac{v \sin \theta}{g}$$

Total time of flight:

$$T = \frac{2v \sin \theta}{g}$$

Horizontal range:

$$R = \frac{v^2 \sin 2\theta}{g}$$

General equation of trajectory:

$$y = x \tan \theta - \frac{gx^2}{2v^2 \cos^2 \theta}$$

In geology the thickness of a rock stratum having a level outcrop may be approximately determined by finding the horizontal length of the outcrop, d, and the angle of dip, θ. The thickness of the stratum, W, is then given by the formula $W = d \sin \theta$.

The construction of a sundial, while not of great practical importance, involves an interesting application of trigonometry. The positions of the hour lines on the dial are determined by the formula $\tan h = \sin l \tan t$, where h is the angle between any particular hour line and the meridian, l the latitude at the point of observation, and t the hour angle which, in degrees, is 15 times the number of hours between noon and the hour corresponding to that hour line.[2]

[1] The Chevrolet Motor Company has made available a series of interesting posters depicting symbolically and stating explicitly this and other great mechanical and mathematical principles involved in automotive engineering. These charts are available without cost and are discussed at length in an article entitled "Mathematics of the Automobile," *The Mathematics Teacher*, **31** (1938), 209–215.

[2] See C. N. Shuster and Fred L. Bedford, "Field Work in Mathematics" (New York: American Book Company, 1935), pp. 88–89; for a much more complete

Most astronomical work and long-distance navigation and aviation require the application of spherical trigonometry for the determination of courses, distances, and positions. For comparatively short courses ("plane sailing" and short-distance flying) the errors due to the earth's curvature are relatively small and do not affect seriously the computations based upon the formulas of plane trigonometry. Thus, by noting the initial latitude (L), the course or direction (C), and the distance (D) run on that course, one can determine readily the *departure* (distance traveled eastward or westward), the *difference in longitude* (indicated by $D\,Lo$), the *difference in latitude* (distance in nautical miles made good northward or southward, and indicated by l), and the ship's position at the end of the run. If the course angle C is taken to be that angle measured clockwise (looking down) from north around to the course line, and if all distances are given in nautical miles, then these formulas hold:[1]

$$\text{Difference in latitude} = D \cos C$$
$$\text{Departure} = D \sin C$$
$$D = (\text{difference in latitude})(\sec C) = (\text{departure})(\csc C)$$
$$C = \arccos \frac{\text{difference in latitude}}{D} = \arcsin \frac{\text{departure}}{D}$$
$$\text{Difference in longitude} = (\text{departure})(\sec L) = D\,(\sin C)(\sec L)$$

In these days of rapid expansion of aviation many young people will be interested in air navigation. Trigonometry plays an important part in this, and those who pilot or navigate planes will need to be concerned with the mathematics of air navigation. Two important problems are the determination of the wind-correction angle and the determination of ground speed. These problems may be solved by scale drawing, or they may be solved by application of the law of sines from plane trigonometry. If, for example, the wind-correction angle is designated by x and the angle between the wind direction and the true course by y, then

$$\frac{\text{Wind velocity}}{\text{Air speed}} = \frac{\sin x}{\sin y}$$

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discussion of this problem see LaVergne Wood and Frances Mack Lewis, The Mathematics of the Sundial, *The Mathematics Teacher*, **29** (1936), 295–303.

[1] For further discussion of plane sailing and allied topics see Shuster and Bedford, *op. cit.*, pp. 68–82, 94–97. See also C. H. Butler and F. L. Wren, "Trigonometry for Secondary Schools" (Boston: D. C. Heath and Company, 1948), pp. 93–97, or any of several recent books on nagivation.

Thus sin x (and therefore x) may be readily determined, since all the other elements in the formula presumably are known.

If the same notation is used, the formula for determining ground speed is as follows:

$$\frac{\text{Ground speed}}{\text{Air speed}} = \frac{\sin (y - x)}{\sin y}$$

The U. S. Coast and Geodetic Survey publishes graphic charts based on these formulas. By use of these charts the wind-correction angle and the ground speed may be determined very rapidly.[1]

The foregoing miscellaneous illustrations of the use of trigonometry may serve to suggest others. Such applications rarely fail to increase interest in the subject. Teachers will do well to be constantly on the alert for examples of the practical application of the subject. Not only will these enhance the interest of the students, but they will broaden the teacher's horizon and add zest to his teaching.

Exercises

1. Discuss the justification for offering some work in numerical trigonometry in the junior high school. What should be the extent and general nature of this work?

2. Explain in detail how you would develop with a class the meanings of the trigonometric functions.

3. Schultze (see Bibliography) calls the solution of the right triangle the "backbone of practical trigonometry" and says that it constitutes "probably the most important chapter in the entire elementary trigonometry." What are his arguments? Outline his discussion. Do you agree with him? Why, or why not?

4. Find or make up an example of a good field project or laboratory project involving the use of elementary trigonometry. Describe it in detail, and show how it will demonstrate the practical application of trigonometry.

5. What are the main arguments or advantages claimed for offering a course in trigonometry in the senior high school?

6. What do you consider the valid major objectives of a course in trigonometry for the senior high school? Do you find any difference of opinion among authors on this point?

7. What evidence, if any, is available to indicate whether or not the study of trigonometry in high school carries over effectively to subsequent work in college mathematics?

8. Is trigonometry a relatively homogeneous and independent subject, or does it borrow largely from other branches of mathematics?

9. Take the section on the right triangle in any good trigonometry text, and make a careful study of it, analyzing it to find out what concepts, information, and skills of (*a*) arithmetic, (*b*) algebra, and (*c*) geometry are involved.

[1] For a rather full discussion of these problems see John Kinsella and A. Day Bradley, Air Navigation and Secondary School Mathematics, *The Mathematics Teacher*, **32** (1939), 82–86. See also T. C. Lyon, Practical Air Navigation, *Civil Aeronautics Bulletin* 24, or any of several recent books on air navigation.

10. Outline a semester's work in trigonometry for a senior-high-school class, giving the sequence of major topics with approximate time allowance for each and the main subtopics to be studied under each. Assume that you are the supervisor, and, in connection with several of these subtopics, write out the suggestions which you would give to the members of your staff.

11. Do you believe it is better to start with a study of the right triangle and develop the concepts of the trigonometric functions in terms of the right triangle, or would it be better to begin with a study of the general angle and develop and use the concepts of the functions in terms of the general angle from the outset? Give the arguments for and against each of these plans.

12. What specific values may be attained through a study of the graphs of the trigonometric functions?

13. By the method of projection make careful graphs of the six functions of an angle as the angle varies from -180 degrees to $+180$ degrees.

14. Why do so many students fail to get an appreciation of the full significance of the following:

a. The periodic nature of the functions?

b. The multiple-valued nature of the inverse functions as contrasted with the single-valued nature of the functions themselves?

c. The use of line values in connection with the variation of the functions other than sine and cosine?

d. Undefined values of functions at points of discontinuity?

15. What could be done to improve understanding in these areas?

16. Summarize the discussion of the teaching of the functions of the special angles as it is given in this chapter.

17. Would you expect your students to memorize the formulas for restating the functions of angles greater than 90 degrees in terms of functions of angles less than 90 degrees? Why, or why not? If not, how would you have them make these transformations?

18. Make a list of all the trigonometric formulas which, in your opinion, ought to be memorized by students in a class in this subject.

19. What are logarithms? Do they constitute an integral part of trigonometry? Explain clearly their function in connection with the study of trigonometry.

20. Why do students have so much trouble with logarithms? How can the logarithmic work be improved? Would emphasis on scientific notation help promote understanding of logarithms?

21. What instruction or explanation would you give to your students for determining the characteristic of the logarithm of a given number?

22. Explain in detail how you would teach interpolation in connection with logarithms or natural trigonometric functions.

23. What is a slide rule? How does it work, and why? In what way is it related to logarithms?

24. What account should be taken of the approximate nature of the data, tables, and results in problems involving numerical trigonometry?

25. What are the basic principles to be observed in computation with approximate data?

26. What are the relative merits of four-place tables or five-place tables as aids to computation in the solution of trigonometric problems?

27. Should any spherical trigonometry be taught in the high-school course? Give arguments both for and against this.

28. Give a good explanation of radian measure as you would explain it to a class. Illustrate by the use of several special angles.

29. Why should students be taught radian measure and the radian notation?

30. Is it desirable to give work in proving identities and solving trigonometric equations to high-school students? Compare the views of different authors on this before you draw a conclusion.

31. Explain the fundamental distinction between proving trigonometric identities and solving trigonometric equations. Why is it not legitimate in proving identities to use the algebraic laws of the equation and operate as if the identity to be proved were an equation?

32. Illustrate and explain the "function hexagon" described in this chapter.

33. Compare several textbooks in plane trigonometry, and select the one which seems to you to be most suitable for high-school use. Select also the one which you regard as most suitable for use in college classes. Give your reasons for selecting these particular textbooks.

34. In what particular ways should textbooks in trigonometry be improved to make them more suitable for high-school classes?

35. Make a list of examples illustrating the fact that an understanding of the principles of elementary trigonometry is sometimes needed for clear understanding of matters of general interest. Give bibliographic references for all examples in your list which are not original with you.

36. Develop the formulas for sin $(A + B)$, cos $(A + B)$, and tan $(A + B)$. Show that the corresponding double-angle formulas are but special cases of these.

37. Prepare an outline, as for a discussion with your class, of the role of trigonometry in navigation and aviation.

38. Take five recent textbooks in trigonometry, and compare them with respect to their treatments of trigonometric identities and equations. Which one do you like best? Why?

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CHAPTER XVIII

THE TEACHING OF CALCULUS

Except for a few scattered articles in one or two professional magazines, very little has been written on the teaching of calculus. This is doubtless due to the fact that calculus has always been regarded as essentially a college subject. For various reasons efforts looking toward the improvement of the teaching of mathematics have been very largely confined to the courses of the elementary and the junior and senior high schools. The journals dealing with college mathematics give expositions of subject matter but rarely concern themselves with pedagogical considerations.

That this should be so is not hard to understand. In general, college students are presumed to be intellectually more mature and more highly selected than students in the junior or senior high schools and may reasonably be expected to assume more responsibility for the mastery of their work. The subject matter which they study is increasingly rigorous and exacting. Most important of all, their instructors have, as a rule, far more extensive academic training than the teachers in the lower schools. Their viewpoints are more specialized, and they tend to be more interested in the logical aspects of their courses than in the manner in which the subject matter is most effectively learned.

For these reasons, among others, the literature on the teaching of mathematics has included very little discussion of the teaching of college mathematics.

On the other hand, the mortality among students of college mathematics is relatively high. It is probably higher than it needs to be, and higher than it would be if instructors would give more attention to the careful development of the various topics, to diagnosis and remedial teaching, and to a study of the particular difficulties which are characteristically encountered by their students. Calculus, in particular, has acquired the reputation of being extremely difficult, and many students approach it with fear and trembling. That it involves substantial difficulties nobody would deny, but most of these can be considerably mitigated by good teaching.

Calculus in the High School. Since the turn of the century various leaders in mathematical education have advocated that elements of the calculus be introduced in the last year of the senior high school. It was not contemplated that this work should comprise the complete course in differential and integral calculus usually offered as the second year of college mathematics. Obviously such a course would require more adequate mastery of algebra, trigonometry, and analytic geometry than could be reasonably expected at the end of the eleventh grade. Rather, the idea was that just as there are certain parts of trigonometry that are simple enough to be successfully understood by junior-high-school students, so there are certain parts of calculus which can be understood by twelfth-grade students, even if they do not have all the prerequisites for a complete systematic course.

This position was well stated in the suggestions of the National Committee on Mathematical Requirements regarding calculus in the high school.

In connection with the recommendations concerning the calculus, such questions as the following may arise: Why should a college subject like this be added to a high school program? How can it be expected that high school teachers will have the necessary training and attainments for teaching it? Will not the attempt to teach such a subject result in loss of thoroughness in earlier work? Will anything be gained beyond a mere smattering of the theory? Will the boy or girl ever use the information or training secured? The subsequent remarks are intended to answer such objections as these and to develop more fully the point of view of the committee in recommending the inclusion of elementary work in the calculus in the high school program.

By the calculus we mean for the present a study of *rates* of change. In nature all things change. How much do they change in a given time? How fast do they change? Do they increase or decrease? When does a changing quantity become largest or smallest? How can rates of changing quantities be compared?

These are some of the questions which lead us to study the elementary calculus. Without its essential principles these questions cannot be answered with definiteness.

The following are a few of the specific replies that might be given in answer to the questions listed at the beginning of this note: The difficulties of the college calculus lie mainly outside the boundaries of the proposed work. The elements of the subject present less difficulty than many topics now offered in advanced algebra. It is not implied that in the near future many secondary school teachers will have any occasion to teach the elementary calculus. It is the culminating subject in a series which only relatively strong schools will complete and only then for a selected group of students. In such schools

there should always be teachers competent to teach the elementary calculus here intended. No superficial study of calculus should be regarded as justifying any substantial sacrifice of thoroughness. In the judgment of the committee the introduction of elementary calculus necessarily includes sufficient algebra and geometry to compensate for whatever diversion of time from these subjects would be implied.

The calculus of the algebraic polynomial is so simple that a boy or girl who is capable of grasping the idea of limit, of slope, and of velocity, may in a brief time gain an outlook upon the field of mechanics and other exact sciences and acquire a fair degree of facility in using one of the most powerful tools of mathematics, together with the capacity for solving a number of interesting problems. Moreover, the fundamental ideas involved, quite aside from their technical applications, will provide valuable training in understanding and analyzing quantitative relations—and such training is of value to everyone.[1]

With reference to the content of such a course as was contemplated, the Committee stated

The work should include:

a. The general notion of a derivative as a limit indispensable for the accurate expression of such fundamental quantities as velocity of a moving body or slope of a curve.

b. Applications of derivatives to easy problems in rates and in maxima and minima.

c. Simple cases of inverse problems; *e.g.*, finding distance from velocity, etc.

d. Approximate methods of summation leading up to integration as a powerful method of summation.

e. Applications to simple cases of motion, area, volume, and pressure.

Work in the calculus should be largely graphic and may be closely related to that in physics; the necessary technique should be reduced to a minimum by basing it wholly or mainly on algebraic polynomials. No formal study of analytic geometry need be presupposed beyond the plotting of simple graphs.

It is important to bear in mind that, while the elementary calculus is sufficiently easy, interesting, and valuable to justify its introduction, special pains should be taken to guard against any lack of thoroughness in the fundamentals of algebra and geometry; no possible gain could compensate for a real sacrifice of such thoroughness.

It should also be borne in mind that the suggestion of including elementary calculus is not intended for all schools nor for all teachers in any school. It is not intended to connect in any direct way with college entrance requirements. The future college student will have ample opportunity for calculus

[1] Report of the National Committee on Mathematical Requirements, "The Reorganization of Mathematics in Secondary Education" (Boston: Houghton Mifflin Company, 1923), pp. 57–59.

later. The capable boy or girl who is not to have the college work ought not on that account to be prevented from learning something of the use of this powerful tool. The applications of elementary calculus to simple concrete problems are far more abundant and interesting than those of algebra. The necessary technique is extremely simple.[1]

This Report of the National Committee gave some impetus to the inclusion of certain ideas and work from calculus in the high school. The attempts, however, were scattered and sporadic, and, while a certain amount of interest and success was reported in some instances, there appears to have been little general enthusiasm for it. This may have been due, in part at least, to the lack of suitable textbooks. Whatever may have been the reason, calculus as a subject never has attained a position of any prominence in the senior high school. Indeed, it is given little prominence among the mathematical branches listed in the grade placement chart in the later (1940) Report of the Joint Commission, although the Report does suggest, in connection with algebra in the twelfth grade "introducing study of differentiation, limited to polynomials, with applications to slopes, maxima and minima, rates of changes, velocity, accelerations, and related problems."[2]

From these indications it appears that calculus as a subject probably is not destined to find a place in the high-school curriculum but will in all likelihood remain distinctively a college subject. There are numerous opportunities in high-school work, however, in connection with algebra, geometry, trigonometry, graphical work, and physics to bring about the preliminary development of fundamental concepts which will be needed in the subsequent study of calculus. Much can be done to illustrate such concepts as rates of change, maxima and minima, slopes, independent and dependent variables and functions, limits, and approximate methods of summation. If time permits, it is easily possible even to develop the notion of a derivative and rules for differentiating simple algebraic functions. Simple applications involving time rates, maxima and minima, etc., will add interest to this work, as will the mention of how calculus may be used as a powerful means of summation to find areas, volumes, pressures, and the like. The

[1] *Ibid.*, pp. 54–55.

[2] Joint Commission of the Mathematical Association of America, Inc., and the National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, *Fifteenth Yearbook of the National Council of Teachers of Mathematics* (New York. Bureau of Publications, Teachers College, Columbia University, 1940), pp. 97–98.

student who approaches his college calculus equipped with such a background will certainly find the work easier and more meaningful than it would be otherwise. The high-school teacher of mathematics who is able to give his students the advantage of such a springboard will have done much to enrich their conception of the subject and to ensure their success in it.

In the remainder of this chapter the teaching of various parts of the calculus will be considered primarily from the standpoint of the systematic course in calculus in the second year of the junior college.

Fundamental Concepts Should Be Emphasized. Any thorough mastery of the calculus has two fundamental aspects, *viz.*, (1) the understanding of the basic concepts that are involved in the development of the subject and familiarity with the many special formulas that are needed and (2) the acquisition of ability and ready facility in the use of these special formulas and methods. Both are fundamental to a balanced mastery of the subject.

It is possible for an individual to acquire a high degree of skill in the latter aspect of the work without understanding much about what he is doing or why he is doing it. Indeed this situation is often found. The relatively enormous amount of formal work that must be done begets a tendency on the part of instructors to emphasize this aspect often to the detriment of understanding. Many students acquire high proficiency in performing difficult feats of formal differentiation or integration without having any clear conception of the meaning of a derivative or an integral or of the real significance of what they are doing. Proficiency in this mechanical aspect of the work, although it is indispensable, tends to produce complacency on the part of both the student and the teacher. This is not unnatural, but it is unfortunate. Such a one-sided development indicates, not a balanced mastery of the subject, but rather mere proficiency in rote performance. It is precisely the sort of thing against which so much criticism of the teaching of elementary algebra has been directed. The reason why we hear so little of this criticism leveled at the teaching of calculus is simply because the matter of improving the teaching of college mathematics has as yet received too little attention, at least in the way of published suggestions.

It is true that most of the actual work of differentiation and integration must be carried on by formal methods and that these must be learned and mastered by the students. At the same time, these formal methods and devices can be developed in such a way as to give them meaning, and this should be regarded as a major responsibility of the

instructor. The student who gains a real understanding of the meanings of function, variation, increment, limits, infinitesimals, continuity, derivative, rates, maxima and minima, of an integral as an antiderivative and as a summation, of indefinite and definite integrals, and of other basic concepts will derive a far richer experience and a far more adequate basis for further work in either applied or theoretical mathematics than the student who works by rote alone.

The Teaching of Variables, Limits, and Infinitesimals. Variables, limits, and infinitesimals are primary concepts in calculus, and serious effort should be made to ensure that the students will have a good understanding of their meanings. Normally the students will have gained considerable familiarity with the meanings of "function" and "variable" through their previous work in trigonometry and algebra. Care should be taken that variables and constants are thought of as symbols and not as quantities.

A *variable* is a mathematical symbol which represents any one of a set of values called its "range."

A *constant* is a mathematical symbol which represents the same value throughout the process of a particular discussion.

The student should become familiar with the concept of a range of values and with the nature of, and distinction between, *arbitrary constants* (*parameters*) and *absolute constants.*

It is important that a clear idea of the distinction between dependent and independent variables be established and that the concept of function be definitely associated with the former.

When two or more variables are so related that the value of any one variable depends upon the values of the others, there is said to be a functional relationship that exists between the variables.

In spite of the presumption of familiarity with these notions, it is advisable at the beginning of the course in calculus to review them, giving numerous and varied illustrations involving both algebraic and transcendental functions. It is not anticipated that students will have much difficulty in establishing these concepts, but as a means of ensuring understanding, some practice should be given in identifying independent variables and dependent variables and in building tables and making graphs to illustrate the variation of each of these and the relationship of functions to their independent variables. The student should also become thoroughly familiar with the generalized functional notation. He should learn to recognize $f(x)$ in all its variations and with all its implications.

The concepts of a limit and an infinitesimal and the application of

these concepts are vital to an understanding of the calculus and are indeed the foundation stones upon which the calculus is built. Because of logical difficulties formal treatment of the subject of limits is usually omitted from high-school mathematics. At most, this subject is sometimes taken up in a sort of intuitive way. Evidence of this is found in the common omission of incommensurable cases from the textbooks in elementary geometry. However, in the study of calculus it is very important that there be a careful treatment of limits and infinitesimals.

The student's previous mathematical study has been built entirely upon the elementary principles of algebra and plane geometry. The notion of a function approaching a limit has appeared only in occasional problems. In contrast with this state of affairs, throughout both Differential and Integral Calculus, the limit concept appears as a fundamental principle. This one feature, more than any other, sets the subject of Calculus apart from the student's previous field of study and requires that he enlarge his group of mathematical operations and become familiar with new methods of approach to a mathematical problem.[1]

The difficulty which students most often experience in this connection is not so much in acquiring intuitive concepts of limits and infinitesimals as in understanding their technical definitions. The concept of a limit can be illustrated by various situations drawn from elementary geometry or algebra. Thus let there be a polygon inscribed in or circumscribed about a circle. As the number of sides of the polygon is increased, the difference between the length of its perimeter and the circumference of the circle obviously becomes less than it was at the outset. As this process is indefinitely continued the difference between these two lengths becomes and remains less than any value we may care to assign; *i.e.*, the difference approaches zero as a limit. In other words, the length of the perimeter of the polygon becomes always more nearly equal to the circumference of the circle; the one approaches the other as a limit.

Again, let there be a series of terms of the form $1/2^n$ where $n = 0$, 1, 2, 3, 4, etc. The first term of this series is 1, and, if successive terms are added, the sum of the series becomes

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots .$$

The addition of successive terms brings the sum nearer and nearer to 2. No finite number of terms can make the sum equal to 2, but by taking n sufficiently large, the difference between the sum of the series and 2 can be made as small as desired. Thus we say that, as

[1] From Joseph Vance McKelvey, "Calculus" (New York: The Macmillan Company, 1937), p. v. By permission of The Macmillan Company, publishers.

the number of terms is indefinitely increased, the "sum" approaches 2 as a limit.

The fundamental idea involved in these illustrations is that of a variable difference, diminishing progressively toward zero, between the value of the varying function and the constant which it approaches as a limit. This idea should be emphasized in all the illustrations that are used, because it is only when this is well understood that the somewhat technical definition of a limit, as it is usually given, takes on real meaning. In general, when a variable or function approaches zero as a limit, it is called an "infinitesimal." It must be made clear that an infinitesimal is not merely "a very small quantity," but is a *variable* quantity which can be made smaller than any previously assigned value, no matter how small this value may be; *i.e.*, it approaches zero as a limit. It was the failure to sense this distinction which prevented the ancients from understanding the real nature of continuous variation, and it is precisely the conception of this distinction which enables us to understand the nature of continuity, infinitesimals, and limits, which are the very taproots of calculus.

With this concept of an infinitesimal clearly in mind, the limit of a variable may now be technically defined as follows:

If x is a variable and a is a constant and if it is true that, as x varies, it takes on values such that $|x - a|$ becomes and remains less than h, where h is an arbitrarily small positive quantity, then x is said to approach a as a limit.[1]

Evidently this definition can be extended to define the limit of a function. Thus,

If $f(x)$ is a function of x and a and A are constants and if, for any arbitrarily chosen small positive quantity d, there exists another small positive quantity h such that, when $|x - a| < h$, it is true that $|f(x) - A| < d$, then $f(x)$ is said to approach A as a limit as x approaches a as a limit.

Every effort should be made to make these definitions meaningful to the students. To this end graphic and numerical illustrations should be used in the explanations, which should be more than perfunctory. These definitions are of great importance in that they provide the analytical means of defining the important property of continuity of a function or a variable as well as the indispensable concept of a derivative.

The students should be made familiar with the customary notation used in connection with limits. They should also become familiar with certain theorems concerning limits. These may be, and usually

[1] The symbol $|x - a|$ is to be read "the absolute value of $x - a$" or "the numerical value of $x - a$." It is to be interpreted as meaning the difference between x and a without regard to sign.

are, given without proof, but their statements should be accompanied by ample illustrative explanation to ensure that their meanings are really understood by the students.

Teaching the Meaning of a Derivative and a Differential. The late Prof. Louis Ingold once said that whoever really understands the meaning of a derivative has learned the most of calculus. He had no reference, of course, to operational facility, but rather to the fact that an understanding of the derivative underlies the conception of what calculus is all about. The importance of the concept of a derivative and the associated concept of a differential cannot be too greatly emphasized. Therefore, if calculus is to be taught with the idea of giving the students something more than a rote mastery of its techniques, it is highly important that every effort be made to give them at the outset a clear understanding of these two fundamental concepts.

A derivative might be thought of as the instantaneous rate of change of a function with respect to its independent variable. The usual method of defining the derivative in terms of increments, ratios, and limits, with variation in details, is substantially the same in nearly all textbooks. Like many careful definitions, however, its meaning is difficult for many students to apprehend. There seems to be something mysterious about the process of passing to the limit and of determining the limiting value of the ratio of the increment of the function to the increment of the independent variable when the latter approaches zero as a limit. The student reasons that, if the one increment approaches zero, the other will also approach zero, and the ratio will be reduced in this way to the form 0/0 which, in form, appears meaningless due to the division by zero.

In order to clarify this concept, it is absolutely necessary to make the student understand that the increment of the dependent variable can and must be expressed *in terms of the independent variable and its increment*, and that the ratio of the increment of the dependent variable to the increment of the independent variable thus becomes a function of the increment of the independent variable. Furthermore, it should be emphasized that it is the limit of this ratio that is sought and not the ratio of the limits of the numerator and denominator as separate functions. Thus let $y = f(x)$ be a continuous function of x. If x takes on an increment Δx, then y will also take on an increment Δy, such that $y + \Delta y = f(x + \Delta x)$, or $\Delta y = f(x + \Delta x) - y$, or

$$\Delta y = f(x + \Delta x) - f(x).$$

In this way Δy is expressed entirely in terms of x and Δx. Then the

ratio $\frac{\Delta y}{\Delta x}$ becomes $\frac{f(x + \Delta x) - f(x)}{\Delta x}$, which is reducible in general to a determinate form and should be so reduced *before the limit is taken.* Graphic representations such as are given in nearly all textbooks help greatly in giving concreteness to the concepts of these increments and ratios, and full advantage should be taken of this means of illustrating and clarifying the concepts.

The notation also is confusing to the students, who are puzzled about such questions as these: What is the distinction between Δy and dy and between Δx and dx? Why is it that Δy is not in general

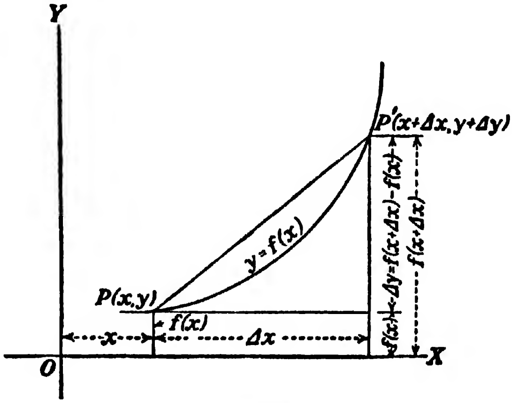

FIG. 70.

equal to dy, while Δx is in general equal to dx? How does Δy become dy? What does dy mean? If dy/dx is defined as the limit of $\Delta y/\Delta x$ as Δx approaches zero, why can it not be thought of as a quotient? Are dy and dx infinitesimals? Are Δy and Δx infinitesimals? What is the distinction between an infinitesimal and an increment, and between an increment and a differential?

It is recommended that the notation dy/dx not be used for the derivative until after the concept of differential is developed. A good notation for the derivative of y with respect to x is $D_x y$; similarly the derivative of $f(x)$ with respect to x would be $D_x f(x)$. There are other good notations that do not introduce the confusion of thinking of a quotient where there is no quotient. It seems probable that a good deal of the confusion and uncertainty concerning the distinction between increment, differential, and derivative could be avoided if, contrary to custom, at least an intuitive notion of the meaning of a differential were to be developed along with the discussion of the

derivative. This can easily be done by means of geometric considerations and an arbitrary definition. In the accompanying figure suppose that P is some point (x, y) on the curve $y = f(x)$. If x takes on an increment Δx, then y must take on an increment Δy whose value depends upon the value of Δx. Therefore the ratio $\Delta y/\Delta x$ would give the average rate of change of y with respect to x over the interval Δx, but not the instantaneous rate of change at the beginning of the interval. The instantaneous rate of change at any point on the curve is given by the slope of the curve at that point, and the slope of a curve at a given point is defined as the slope of the tangent to the curve at that point. Thus the derivative of a function at a point, which is defined to be $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ and which gives the instantaneous rate of change for the corresponding value of the dependent variable, also gives the slope of the tangent to the curve at that point.

If we draw PT tangent to the curve at $P(x, y)$, the distance dy, as shown in the accompanying figure, represents the increment which y would have taken if its rate of change with respect to x had become constant exactly at the point $P(x, y)$. As P' moves along the curve to the position of P (Fig. 71), both Δx and Δy approach zero as a limit. If the ratio $\Delta y/\Delta x$ represents the average rate of change for any of the intervals Δx, and the ratio dy/dx the rate of change for the tangent line PT, it is evident that $\Delta y/\Delta x$ is changing in value as Δy and Δx approach zero, while dy/dx is remaining constant. For each new position of P', the Δx and dx are the same in value, while Δy and dy differ. Furthermore, $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}$. By agreement we shall call dx, which is the same as Δx, *the differential of* x, and we shall call dy *the differential of* y corresponding to dx. We have shown that the ratio dy/dx of the differential dy and dx is the same as the *derivative of* y *with respect to* x. The value of this ratio will vary according as P takes different positions on the curve, but for any particular position of P (*i.e.*, for any given value of x) the value of the derivative dy/dx remains constant and is entirely independent of the value of dx.

Now, if we wish to let dx become an infinitesimal and approach zero as a limit, dy will also become an infinitesimal and approach zero as a limit, but the ratio dy/dx remains the same, even in the limit. However, it is no longer necessary to regard the differentials dy and dx as infinitesimals unless we so wish. They may be regarded simply as tangible, finite increments, dx being any arbitrary increment of x, and dy being the increment which y *would take on* if the slope of the curve

(or the rate of change of y with respect to x) had become constant at the point $P(x, y)$.

It is believed that an approach to the concept of the derivative along with this intuitive notion of the differential as a finite, measurable quantity rather than an infinitesimal would dispel much of the mystery and uncertainty so often associated with the meaning of derivatives and differentials. The student's understanding will be

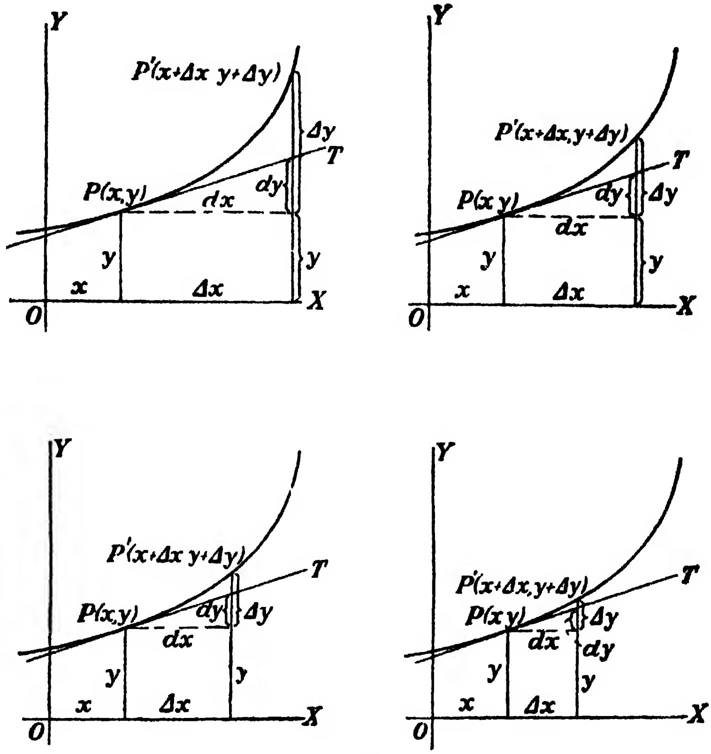

FIG. 71.

strengthened if he is asked to take various curves, to select arbitrary values and arbitrary increments for the independent variable, and by drawing and actual measurement to determine dx, $f(x)$, $f(x + \Delta x)$, Δy, and dy. Then by actual division, dy/dx, he can get approximations to the values of the derivatives To develop the rules for differentiation, of course, the usual analytic procedure will need to be employed. However, if a preliminary approach such as has just been described can make this analytic procedure more meaningful to the student, it will be exceedingly worth while.

Teaching the Rules for Differentiation. It will soon become apparent to the student that the process of finding the derivative of a func-

tion by the graphical method gives only approximations at isolated points and does not give any general expression for the derivative. He will also find that the detailed application of the formula

$$D_x y = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

while it does give a general and exact expression for the derivative, is a clumsy and laborious process. The practical limitations of time demand that he learn and use the special rules or formulas which are available for differentiating various kinds of functions.

There are two points of view with regard to the introduction, learning, and use of these rules for differentiation. Considered solely from the standpoint of logical sequence and mathematical consistency, it would seem necessary that each of these formulas should be completely developed and proved before being used. This is the position taken by the authors of most textbooks on calculus. On the other hand, there are some authors and many teachers who hold that it is a sounder and more economical pedagogical procedure to introduce without proof certain of these rules in which the meaning is clear and to let the students use these empirically for the time being, the proofs being reserved until later, and special attention being given at the time only to the matter of making sure that the concepts involved are understood by the students. Thus Prof. Huntington writes:

Now the simplest function is the polynomial,

$$y = A + Bx + Cx^2 + Dx^3 + \cdots.$$

I should begin the second day, therefore, by stating, without proof, the rule for differentiating a polynomial, namely:

$$dy = (B + 2Cx + 3Dx^2 + \cdots)\, dx.$$

Equipped with this rule, we can proceed at once to solve a multitude of problems in maxima and minima, which serve better than anything else to convince the student that here is a new tool which is mighty convenient to have at hand. . . .

The study of the polynomial naturally leads to problems involving the quotient of two polynomials, and to problems involving the solution of a quadratic equation. In order to handle such problems, it is well to introduce at this point the statement, without proofs, of the general rules for differentiating a sum, a product, and a quotient, and also the special rules

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2}\,dx, \qquad d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}$$

in which x may be the independent variable, or itself a function of some other variable. . . .

If the meaning of a theorem is clear, as in the case of the rules for differentiation, the formal proof may often be postponed to advantage, but . . . the meaning of a new concept cannot safely be postponed . . . .[1]

With regard to the subsequent proofs of the rules for differentiation, Prof. Huntington later writes:

If the time-saving program above outlined has been followed, the student should now have a good working knowledge of the properties of all the elementary functions including the rules for differentiation. If he has a spark of scientific curiosity, he will be interested to know how these rules, whose utility he has come to appreciate, were ever discovered, and how we know they are true. . . .

To indicate how much time can be saved, by taking up the proofs in the most effective order, I may say that the proofs of all the rules for differentiation can easily be disposed of in two classroom periods. First establish the rule for the sine and the rule for the logarithm, for the general case of a function of a function. Then the rules for kx, x^n, e^x, uv, u/v, $\cos x$, $\tan x$, and the inverse functions follow by a turn of the hand.[2]

This will doubtless be regarded by many teachers as an overstatement and an oversimplification of the situation, and it may be that it is. There are also those who will object to it on the grounds that it lacks the sequential rigor of the prove-as-you-go plan, though the basis for this objection may be more apparent than real. However, regardless of whether or not one subscribes wholly to the suggestions made by Prof. Huntington, it seems probable that whatever sacrifice of mathematical order they would entail might be largely compensated by increased interest and operational efficiency.

If the student is to use the differentiation formulas with efficiency and dispatch, he will need to memorize them and memorize them thoroughly. As soon as a rule is given, whether with or without proof, it should be applied immediately, and considerable practice should be given in finding derivatives of functions under this rule. This will do a great deal toward enabling the student to fix the rule in his mind. However, as the number of new formulas increases, the difficulty of keeping them straight becomes greater, and the student will in general have to resort to actual and thorough memorization. Careful inspection of a tabulated list of the formulas will perhaps enable the student to discover certain relationships among them and to set up various mnemonics to aid in remembering them and to avoid confusing them with each other.

[1] Edward V. Huntington, Teaching the Calculus, Four Papers on the Teaching of Mathematics, *Bulletin* 19 (Lancaster, Pa.: Society for the Promotion of Engineering Education, 1932) pp. 39, 40, 41.

[2] *Ibid.*, pp. 46–47.

Some of the formulas are special cases of others. For example, $D_x(cu)$ is a special case of $D_x(uv)$, $D_x(e^u)$ is a special case of $D_x(a^u)$, and other similar instances may be found. Again certain similarities may be found among the derivatives of certain related inverse functions. The derivative of the arccos x is minus the derivative of the arcsin x; that of the arccot x is minus that of the arctan x; and that of the arccsc x is minus that of the arcsec x.

These and perhaps other relationships that might be found among the formulas will reduce the amount of sheer memorizing that will need to be done, and the very search for such mnemonic devices is itself an excellent exercise in familiarizing the students with the formulas.

Some Critical Points in Developing the Proofs of Certain Rules for Differentiation. We have already discussed the matter of teaching the meaning of a derivative and of a differential. A clear concept of the meanings of these terms is prerequisite to any real understanding of the proofs of the formulas for the elementary derivatives. These concepts, in turn, involve an understanding of the meanings of increments, limits, and infinitesimals, which have also been discussed. It is probable that teachers, in developing the formulas for derivatives, are inclined to be too generous in their assumptions regarding the students' mastery of the meanings of these concepts.

The general method of finding a derivative needs to be stressed. Its almost mechanical form and the one-two-three order of its steps, together with the reasons for this order, need to be explained and illustrated, not once, but numerous times, until the students are able to apply it independently and with facility to simple functions. It is easy for them to follow and verify, step by step, the illustration of it, but it is by no means equally apparent to them *why* the particular steps are taken in that particular order. Consider the typical order in the general development:

(1) $y = f(x)$

(2) $y + \Delta y = f(x + \Delta x)$

(3) $\Delta y = f(x + \Delta x) - y = f(x + \Delta x) - f(x)$

(4) $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

(5) $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = D_x y = D_x f(x)$

(by definition)

From a synthetic standpoint it is perfect, each step being a justified consequence of the preceding one. However, it cannot be said that each step *suggests* the following one, and herein lies the rub. The

steps and their order suggest themselves only if one has in mind the end toward which he is working. It must be the task of the teacher to point out what this goal is and how and why this particular sequence of steps does lead to it. It is well to carry along this general synthesis with the illustrations of its application to particular functions, pointing out the parallel development, and repeatedly calling attention to the reason for performing each step in its particular place.

The explanation may be somewhat as follows: Let us assume that we have given a variable y which is a function of x. Our task is to find the derivative of y with respect to x, or to find D_xy. By definition D_xy means $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. We must therefore find some way of getting an expression $\Delta y/\Delta x$ in terms of x and Δx in order that we may determine its limit and thus find the derivative.

General Case	Example
$y = f(x)$	$y = 3x + 7$
$y + \Delta y = f(x + \Delta x)$	$y + \Delta y = 3(x + \Delta x) + 7$

Since we want to get an expression $\Delta y/\Delta x$, we must first get an expression for Δy itself. This we can do by subtracting y from each member of the above equation.

$\Delta y = f(x + \Delta x) - y$	$\Delta y = 3(x + \Delta x) + 7 - y$

We may now substitute for y its value $f(x)$ (or $3x + 7$) and express Δy entirely in terms of x and Δx.

$\Delta y = f(x + \Delta x) - f(x)$	$\Delta y = 3(x + \Delta x) + 7 - (3x + 7)$
	$= 3x + 3\,\Delta x + 7 - 3x - 7$
	$= 3\,\Delta x$

Now in order to get an expression for $\Delta y/\Delta x$, we must divide both members of the equation by Δx.

$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$	$\frac{\Delta y}{\Delta x} = \frac{3\,\Delta x}{\Delta x} = 3$

Finally, since we want D_xy, and since this means $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, we must take the limit of $\Delta y/\Delta x$ as Δx approaches zero as a limit.

$D_xy = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$	$D_xy = \lim\limits_{\Delta x \to 0} \frac{3\,\Delta x}{\Delta x}$
$= \lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$	$= \lim\limits_{\Delta x \to 0} 3(1)$
	$= \lim\limits_{\Delta x \to 0} 3 = 3$

Several illustrations carried through in this manner, with specific attention to the order of procedure and the reasons that each step comes in its particular place, will be extremely helpful to the students. It will go far toward giving them a real understanding of the fundamental meaning and method of finding derivatives and of developing the general derivative formulas.

Proof of the Formula for the Derivative of a Logarithm. This is one of the particularly "tough" spots which students encounter. The method is perfectly general, but the proof involves certain facts and relations with which students often lack familiarity, although presumably they will have encountered them beforehand. We shall give a proof of the general formula and then analyze some of its difficulties.

(1) $$\text{Let } y = \log_a u, \text{ where } u \text{ is a function of } x$$

(2) $$y + \Delta y = \log_a (u + \Delta u)$$

(3) $$\Delta y = \log_a (u + \Delta u) - y$$

(4) $$= \log_a (u + \Delta u) - \log_a u$$

(5) $$= \log_a \frac{u + \Delta u}{u}$$

(6) $$= \log_a \left(1 + \frac{\Delta u}{u}\right)$$

(7) $$\frac{\Delta y}{\Delta u} = \frac{1}{\Delta u} \log_a \left(1 + \frac{\Delta u}{u}\right)$$

(8) $$= \frac{1}{u} \cdot \frac{u}{\Delta u} \log_a \left(1 + \frac{\Delta u}{u}\right)$$

(9) $$= \frac{1}{u} \log_a \left(1 + \frac{\Delta u}{u}\right)^{u/\Delta u}$$

(10) $$D_u y = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = \frac{1}{u} \log_a \left[\lim_{\Delta u \to 0} \left(1 + \frac{\Delta u}{u}\right)^{u/\Delta u}\right]$$

$$= \frac{1}{u} \log_a e$$

(11) $$= \frac{1}{u} \cdot \frac{1}{\log_e a}$$

(12) $$D_x y = D_u y \cdot D_x u = \frac{1}{u \log_e a} \cdot D_x u$$

The first four steps involve no particular difficulties, but throughout the rest of the proof there are several places where the development is likely to be hard for students to follow. The first of these trouble spots occurs in step (5). Here the student must learn to recognize the application of the principle $\log m - \log n = \log m/n$.

The principle itself will be familiar, but there is possibility that in this unfamiliar dress the student may fail to recognize it unless it is pointed out by the teacher.

In step (6) the reason for the change of form is not obvious, and perhaps it can be made entirely clear only after the next four steps are given. It may be explained at the time that the aim is to arrive at an expression for e. It must then be recalled that

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

and that the expression $\lim\limits_{\Delta u \to 0} \left(1 + \frac{\Delta u}{u}\right)^{u/\Delta u}$, which appears in step (10) and to which the expression $\left(1 + \frac{\Delta u}{u}\right)$ in step (6) contributes, is precisely of this form.

Step (7) involves no difficulty nor does step (8) except that the reason for the change of form, from the expression in step (7), should be explained.

The transition from step (8) to step (9) involves passing from a product form to an exponential expression. This process should be familiar to the students, but, since the expressions are somewhat unusual, it will probably be necessary, and certainly helpful, to point out the identification with the customary form $m \log n = \log (n)^m$.

In step (10), as has been said, it is extremely likely that some of the students will not recognize that $\lim\limits_{\Delta u \to 0} \left(1 + \frac{\Delta u}{u}\right)^{u/\Delta u} = e$. Although the proof of the existence of this limit is too difficult to be presented at this level of instruction, a satisfactory intuitive justification of the fact that $\lim\limits_{x \to 0} (1 + x)^{1/x} = 2.718 \ldots$, approximately, can be given either algebraically or geometrically.

Substitution of e for this limit leads to the formula of step (11) for $D_u y$. However, since we are required to find $D_x y$ and since u is a function of x, we must make use of the formula for the derivative of a function of a function: $D_x y = D_u y \cdot D_x u$. We may assume that this has been previously developed, so that in passing to step (12) we need only recall this rule and point out its application in the present situation.

Finally, it should be pointed out that we have developed the formula for the most general case: $y = \log_a u$. If we take e as the base (as a

special case of a), then $\log_e a$, in the denominator of (12), becomes $\log_e e$, and the formula itself becomes

$$D_x (\log_e u) = \frac{1}{u} \cdot D_x u$$

since $\log_e e = 1$. This is the form in which it is usually found and in which it is generally used.

We have thus seen that the proof of this formula holds numerous specific difficulties for students. In addition to these it involves also the general difficulty of being somewhat long and of involving numerous transformations and substitutions, the reasons for which are not immediately obvious to the students. Only a few of the better students will be able to dig it all out for themselves. Whether the majority really get it or not will depend largely upon how skillfully and clearly it is explained by the teacher.

The proof of the formula for the derivative of the exponential $y = a^u$ follows easily after the formula for the derivative of a logarithm has been established. Conversely, if the formula for the derivative of the exponential is independently developed first, as can be done, it may be used to simplify the development of the formula for the derivative of a logarithm. The order varies with different textbooks. Either may be used. In general it may be said that the one which is developed first, and independent of the other, will be more difficult for the students than the other which makes use of the first. They depend ultimately upon the evaluation of

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a \text{ or } \lim_{x \to 0} (1 + x)^{1/x} = e.$$

Proof of the Formula for the Derivative of the Sine of an Angle. This is another of the basic derivatives and a proverbial trouble spot for students. The steps in the proof may be given as follows:

(1) Let $y = \sin u$, where u is some function of x

(2) $y + \Delta y = \sin (u + \Delta u)$

(3) $\Delta y = \sin (u + \Delta u) - y$

(4) $= \sin (u + \Delta u) - \sin u$

(5) $= \sin u \cos \Delta u + \cos u \sin \Delta u - \sin u$

(6) $\dfrac{\Delta y}{\Delta u} = \dfrac{\cos u \sin \Delta u}{\Delta u} - \dfrac{\sin u(1 - \cos \Delta u)}{\Delta u}$

(7) $= \cos u \dfrac{\sin \Delta u}{\Delta u} - \sin u \dfrac{1 - \cos \Delta u}{\Delta u}$

$$(8)\qquad D_u y = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = \cos u \left[\lim_{\Delta u \to 0} \frac{\sin \Delta u}{\Delta u} \right] - \sin u \left[\lim_{\Delta u \to 0} \frac{1 - \cos \Delta u}{\Delta u} \right]$$

$$(9)\qquad = \cos u(1) - (0)$$

$$= \cos u$$

$$(10)\qquad D_x y = D_u y \cdot D_x u = \cos u \cdot D_x u$$

The first four steps involve no difficulty whatever. In step (5) there is a substitution to be made, and in steps (6) and (7) there are certain rearrangements of terms and the insertion of the divisor Δu, but these, again, involve no special difficulty, so far as the operations themselves are concerned. There is, however, a question as to *why* these particular substitutions and rearrangements are made, and the reasons should be explained clearly to the students. In order to present these reasons, however, one must recall that it is necessary, in setting up the derivative, to evaluate $\lim\limits_{\Delta u \to 0} \frac{\Delta y}{\Delta u}$. It should be explained that in order to do this we need to use $\lim\limits_{\Delta u \to 0} \frac{\sin \Delta u}{\Delta u}$ and $\lim\limits_{\Delta u \to 0} \frac{1 - \cos \Delta u}{\Delta u}$ since these expressions can be evaluated and their use affords the only way to evaluate $\lim\limits_{\Delta u \to 0} \frac{\Delta y}{\Delta u}$. Indeed, the limiting values of these expressions will already have been determined, but as a matter of refreshing the minds of the students on these points it will be well to review them before substituting them in the formula. As a matter of fact, the two most difficult parts of the whole proof are (*a*) the preliminary establishment of the limiting values of these two expressions, and (*b*) sensing the role which they play in the proof. Step (9) merely involves the substitution of the numerical limiting values for these expressions.

It will be noted that this development gives the derivative of y with respect to u, where u is some function of x. In order to get the derivative of y with respect to x, we must again make use of the relation $D_x y = D_u y \cdot D_x u$. This is done in step (10), which completes the proof.

Successive Differentiation; Maxima and Minima. A few illustrations will suffice to make it clear to the students that in general the derivative of a function of x with respect to x is itself a function of x which may in turn be differentiated and that, by differentiating successive derivatives in this way, there are obtained the so-called "higher derivatives" of the function. The students should be made acquainted

with the various forms of notation for these higher derivatives, and in particular they should be given an interpretation of the second derivative as the rate at which the first derivative is changing, or graphically, as the rate of change of the slope of the curve. That is, the student should come to understand that, if the second derivative is positive at a given point, it means that the first derivative is increasing at that point; a negative value of the second derivative would indicate a decreasing first derivative, and a zero value of the second derivative indicates that the first derivative is neither increasing nor decreasing at that point.

The most important immediate application of the second derivative is in connection with the determination of maximum, minimum, and inflection points. It will be obvious to the student from a consideration of the graphs of functions that the first derivative at a maximum or a minimum point must be zero. The converse does not necessarily hold true, however, and, even if it did, this would not enable one to distinguish between a maximum and minimum. In order to make a certain test of this, the second derivative must also be employed. The student must be shown that, if $f'(x) = 0$ and is decreasing (*i.e.*, if at the same time $f''(x)$ is negative), then a maximum is indicated at that point, while, if $f'(x) = 0$ and is increasing (*i.e.*, if at the same time $f''(x)$ is positive), then a minimum is indicated. If $f'(x) = 0$ and also $f''(x) = 0$, then at that point the function has neither a maximum nor a minimum, but a point of inflection. Careful explanation of these matters should be given and should be accompanied by graphical illustrations to ensure that the students understand them clearly.

The students should be warned against concluding wrongly that $f'(x)$ for a certain x, or at a certain point, will necessarily be zero just because $f''(x)$ is zero. Since $f''(x) < 0$ indicates that the curve is *concave downward* and $f''(x) > 0$ indicates it is *concave upward*, it follows that, at the point where the sense of concavity changes, either $f''(x) = 0$ or $f''(x) = \infty$. This point is called a "point of inflection," or "flex point." That these are not sufficient conditions for a flex point may be seen by examining the curves $f(x) = x^4$ and $f(x) = x^{2/3}$. A simple illustration of how $f'(x)$ and $f''(x)$ may be used in graphing a function is the graph of

$$f(x) = 3x^3 - 2x + 5 \qquad \text{(Fig. 72)}$$

The study of maxima and minima offers a wealth of interesting applications of the theory to geometrical and physical situations. These problems not only give practice in formal differentiation but

also give to the work a high degree of motivation and afford excellent training in interpreting geometric and physical situations and in translating these into the formal language of the derivative.

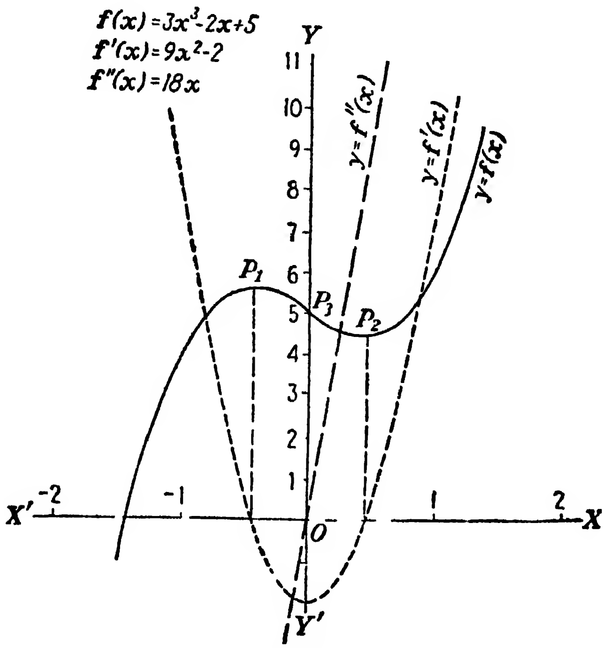

FIG. 72.

The Indefinite Integral; Formal Integration. Up to the present we have considered only matters which are related to derivation and differentiation of functions We must now give some attention to that part of calculus which is concerned with the inverse problem of finding a primitive function of which the given function is the derivative. It should be pointed out to the students that many of the most important applications of calculus give rise to problems of this nature, particularly those involving definite integrals. No student should have any difficulty in understanding the meaning of integration if it is explained simply as the process of antidifferentiation; *i.e.*, if it is clearly pointed out that in differentiation we are given the function and required to find the derivative, while in integration we are given the derivative and required to find the function. The two processes are absolutely inverse to each other, just as division and multiplication, or addition and subtraction. Either one *un*does the other.

In taking up the study of integration, then, the first task of the instructor should be to make this clear to the students so that their

work in integration will not be devoid of meaning. It will help greatly toward achieving this understanding if the instructor and students work out together a few of the fundamental integrals. The finding of a derivative is, of course, a direct process. On the other hand, the finding of an integral form is essentially a matter of trial. In view of the fact that the two processes are exactly inverse to each other, the test for the correctness of an integral is whether or not it can be differentiated to give the original expression.

Using this basic principle, it is possible to take the fundamental formulas for derivatives and, by reversing the reasoning and the notation, arrive at some of the fundamental formulas for integration. A number of these should be set down and tested in this way by the students with the assistance of the instructor. Thus, since $D_x(\sin x) = (\cos x)$, it is seen at once that $\int(\cos x)\,dx = \sin x + C$, because by applying the test for an integral it is apparent that the derivative of $(\sin x + C)$ is $\cos x$. In like manner the integrals corresponding to the other fundamental derivative formulas should be set down, explained, and verified. If this is done, the students can hardly help sensing the relation between derivatives and integrals.

In connection with the development of these integration formulas the student will note the appearance of the constant of integration. The necessity and meaning of the constant of integration must be carefully explained by the instructor. This may be done by use of the theorem "If two functions have the same derivative, they differ only by a constant" and its converse; these may easily be illustrated by examples. It should be pointed out that in many of the applications of integration, the determination of the constant of integration to satisfy initial conditions is of extreme importance.

Use of the Table of Integrals. There are a few of the fundamental integrals with which the student should become perfectly familiar. The lists of these vary in different textbooks, the number usually being between 12 and 25 and depending mainly upon the author's inclination toward generalization or specialization of the forms. The discussion of these forms in each textbook will presumably be consistent with the author's views in this respect, so that the precise number of formulas listed in any case is of less consequence than the fact that there *are* certain of these which the student must know thoroughly. These formulas should be tested by differentiation and should be memorized.

In addition to these fundamental integrals there are many special integrals which are often useful. Textbooks usually contain more or

less extensive lists or tables of these special integrals. Separately published tables, which are much more complete even than those given in the textbooks, are also available. There is no general agreement among instructors as to the extent to which students should use these tables of special integrals. Some feel that, since much of integration is formal anyway, the free use of the tables speeds up the work and allows more to be accomplished without any detriment to the student. Others feel that the student will gain more insight and understanding if he performs most of these integrations for himself by means of the fundamental integrals. Nobody knows just what the optimum is with regard to this question, but it would seem that a middle ground or reasonable balance of these views would perhaps be more defensible than either extreme position. To this end, the suggestion of Prof. Huntington that "no formula in the table should be used until it has at least been verified by 'differentiating back'"[1] seems appropriate.

In using an extensive table of integrals, the student will need to familiarize himself with the way in which the integrals are classified, so that he may readily locate and identify the form corresponding to any given integrand.

In some cases it may be impossible to find in the table a formula which corresponds to the given integrand. In such cases it is sometimes possible to transform the given expression into usable form through such special devices as resolution of the expression into partial fractions, the substitution of a new variable, application of the rule for integration by parts, or use of the reduction formulas. These special devices involve special procedures the reasons for which, and the real significance of which, will probably not be clear to the students unless the underlying principles and considerations are carefully explained by the instructor. In order that the students may appreciate and become familiar with the nature of these devices, ample illustration and explanation of them should be given. In connection with trigonometric substitutions and the transformation of trigonometric expressions into integrable forms, it may be advisable to give again a brief summary and review of certain of the trigonometric identities, notably the addition formulas and the functions of half angles and double angles. The extreme generality of the reduction formulas makes it desirable that illustrative examples be worked out by the instructor to familiarize the students with the precise manner in which these formulas are applied to particular functions, and with the effect which is produced in the integrand through the application of these formulas.

[1] Huntington, *op. cit.*, p. 55.

The Definite Integral. As in the case of the indefinite integral, many students employ the definite integral without understanding clearly its nature and interpretation. The indefinite integral was defined merely as an antiderivative. The definite integral, on the other hand, is to be interpreted in a wholly different way, *viz.*, as a summation of elements having the characteristic form $f(x)\ \Delta x$, or, more precisely, as the limit of the sum of these elements as Δx approaches zero as a limit. The justification for identifying the limit of such a sum with an integral rests, of course, upon the fundamental theorem for definite integrals, and this must be made clear to the students in due time. Here is a case, however, in which the concept of the definite integral may justifiably be developed, explained, illustrated, defined, associated with its characteristic symbolism, and actually used, before proceeding to a proof of the fact that its use is justified.

In general there are three phases to problems involving definite integrals: (1) setting up the element of integration, (2) performing the formal integration, and (3) substituting limits and evaluating the integral. The third of these is mere labor. The second involves the knowledge and ability required to perform the integration correctly. This presumably will have been developed in connection with the work on indefinite integrals. The first phase, setting up the formula for the element of integration, is the part of the work which is likely to cause the students the most difficulty. Since ideas seem to be most readily acquired and assimilated when they are associated with graphic representations, the most effective illustration of the method of setting up the formula for the element of integration is probably in connection with the problem of finding the area under a curve. Here it is apparent that the element of area, ΔA or $ABCD$, is approximately equal to the area of the rectangle $ABED$, this area being given by $y\ \Delta x$, or $f(x)\ \Delta x$. Passing to the differential notation, we have $dA = f(x)\ dx$ which is the characteristic form for the differential of area or the element of integration, whence $A = \int dA = \int f(x)\ dx$. The important thing here is that the element of integration is always of the form $dA = f(x)\ dx$, and the area itself is given as $A = \int f(x)\ dx$, or $F(x)$. If $x = a$, we have $A = F(a)$, and, if $x = b$, we have $A = F(b)$. The area under the curve and *between* the ordinates erected at a and b will evidently be $F(b) - F(a)$, which, as we have seen, would be given by

$$\int_{x=b} f(x)\ dx - \int_{x=a} f(x)\ dx.$$

It is to be noted that the constant of integration would appear in

both of these integrals but would disappear in the subtraction so that it may be disregarded.

The customary notation for the definite integral, $\int_a^b f(x)\,dx$, should be clearly explained, and the student should observe and keep in mind the meaning of every detail of this notation. In particular, the student should be aware of the fact that, for a given function $f(x)$, the value of the definite integral $\int_a^b f(x)\,dx$ depends entirely upon the values of a and b.

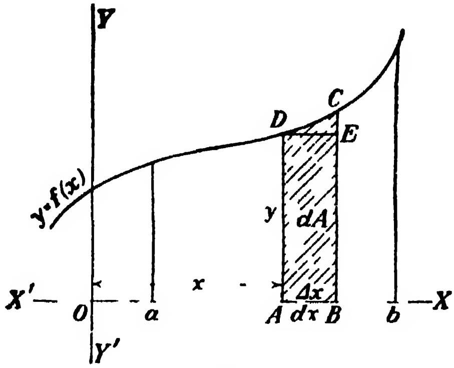

FIG 73.

It should be pointed out to the student that in general the element of integration for any solid can be readily set up provided it is possible to get a characteristic expression for the area of all the sections made by planes parallel to a given plane. If the solid is cut into slices of thickness dh by planes parallel to this plane and if the sections thus made are all similar to each other, then, denoting a typical section by A_s, the element of integration is given by $A_s\,dh$. The meaning of this should be made clear by drawings as should the fact that, if A_s can now be expressed in terms of h, then a definite integral for the volume can be set up. Probably the most common and simplest application of this is in connection with solids of revolution.

Other type forms for the element of integration are encountered in problems involving length of arc, surfaces of solids, plane areas in polar coordinates, moments of mass and inertia, and centroids. Whenever a new type form is to be considered, the instructor should make a special point of explaining and illustrating how the new special form fits into, and in fact derives from, the fundamental concept and definition of an element of integration. If this is done, the student will gradually acquire the ability to interpret problems and set up integrals for himself, and the application of the definite integral will come to have

meaning for him instead of seeming to be merely an assortment of tricks to be learned. If the student once gets a clear concept of the meaning of the element of integration for single definite integrals, the subsequent extension to double, triple, or multiple integrals in general will be plausible and comparatively easy for him.

The foregoing discussion has dealt mainly with the matter of developing meanings and concepts. The illustrations used have dealt with simple functions, and rigorous treatment has not been attempted. This is not to say that rigorous work has no place in connection with the advanced study of the definite integral. On the contrary, if one is to justify completely the use of this important mathematical tool, it is absolutely necessary that a rigorous proof of the fundamental theorem be given.

It should be kept in mind, however, that there is an important distinction between understanding and using this tool and justifying its use. As has been said before, it seems that in this case there is a substantial advantage to be gained by undertaking a thorough development of the concept and application of the definite integral before, or perhaps even without, requiring a rigorous analytical proof of the theorem which justifies its use. Indeed, most elementary textbooks make no pretense of giving a rigorous proof of the theorem but rest the case upon explanations which give understanding and plausibility but which involve a considerable amount of intuition.

All students should be required to understand the line of reasoning which underlies the proof of the fundamental theorem. If the customary geometrical interpretation is used as a basis for the explanation, the problem resolves itself essentially into showing that the area bounded by the curve $y = f(x)$, the x-axis, and the perpendiculars erected to the x-axis at the points $x = a$ and $x = b$ is exactly equal to the area given by the limit of the sum

$$f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + f(x_3)\,\Delta x_3 + \cdots + f(x_n)\,\Delta x_n,$$

as n becomes infinite and each Δx approaches zero as a limit; *i e*, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x_i$ (where $\Delta x_i \to 0$ as $n \to \infty$). This is done by showing

1. That a certain area, say A, is given by $F(b) - F(a)$, where a and b are particular values of x, and $F(x) = \int f(x)\,dx$

2. That $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x_i$ gives *precisely* this same area, A

Thus the students should come to see that the summation process may at any time be replaced by the definite integral, and that the symbols $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$ and $\int_{x_1}^{x_2} f(x)\,dx$ may be regarded as interchangeable, provided the function is single-valued and continuous over the interval.

It should be pointed out that neither the conclusion embodied in the theorem nor the line of reasoning leading to it is limited to, or dependent upon, geometrical considerations, although a geometrical illustration was used and the problem was set up in terms of the summation of elements of area. This is done (1) because the graphic or geometric representation helps to give tangibility and concreteness to a situation otherwise highly abstract, and (2) because functions may be represented graphically and interpreted geometrically even though they refer to nongeometric variables such as forces, heat, work, etc. It should be emphasized that the definite integral may be used to determine any kind of magnitude, provided the characteristic function can be set up in conformity with the requirements stated above.

Improvement of Prerequisite Concepts and Skills. In addition to adding a whole new branch of mathematics to the student's equipment, the study of calculus holds tremendous possibilities for extending and deepening his understanding of the branches previously studied, and the perfection of his skills in these. This is especially the case with reference to algebra, trigonometry, and analytic geometry. It has been said that the place where these subjects are really learned is in calculus. That this is more than a mere figure of speech will be evident from a consideration of the completeness with which the concepts and operations peculiar to these branches underlie and permeate the whole structure of calculus. Algebraic processes find application in innumerable connections throughout the course. Indeed, one of the main problems of formal integration is the algebraic transformation of functions into integrable forms. Trigonometric functions, identities, and transformations are also much in evidence, and a good understanding of analytic geometry is certainly a prime requisite, not only in setting up functions and equations for many of the applied problems, but in giving the student tangible geometric interpretations of the fundamental concepts of calculus.

Functions of many kinds are met with and are made the subject of various investigations and operations. Incidentally it may be noted that calculus gives the student a more comprehensive concept of the nature of a function than he will have been able to get in his previous

study. He will already have gained an understanding of the general meaning of a function and will have made some study of the variation of functions, but in calculus, for the first time, he will make a systematic study of a new aspect of functions, *viz.*, the rate of change of a function with respect to its independent variable.

The student who, in taking up a study of calculus, lacks an adequate background in algebra, trigonometry, and analytic geometry will find himself at a great disadvantage.[1] Indeed, it is not improbable that a large share of the difficulty which students experience in calculus may be directly traceable to inadequate mastery of these prerequisite branches. On the other hand, nowhere could there be found a finer opportunity for well-motivated review and application of their concepts and techniques. Every student should be made conscious of this and should be urged as a matter of enlightened self-interest to put forth every effort to perfect himself in these concepts and techniques. For students who have difficulty, the instructor may be able to perform a real service through helping them to diagnose their troubles and suggesting appropriate remedial exercises. The importance of perfecting the skills and of having ready mastery of the concepts and relationships of algebra, trigonometry, and analytic geometry is a matter which should receive continual emphasis.

Mathematical Rigor in Calculus. The discussion in this chapter has admittedly given special emphasis to the matter of developing concepts and understandings, because it has been felt that in general the teaching of calculus more often falls short in this respect than in any other. Intuitive concepts, however, do not provide sufficiently sound mathematical justification for the conclusions upon which much of calculus is based. Moreover, by the time the student has come through a study of calculus, he should have acquired a feeling for the nature and the necessity for mathematical rigor *as such*. There can be no real appreciation of the nature of mathematical thinking, nor any sound basis for the exploration of higher mathematics, apart from an understanding of what is implied by a rigorous examination of the foundations of mathematics, of the nature of the processes employed, and of the consequences of given conditions.

Calculus stands more or less in the position of a border-line subject with respect to the matter of mathematical rigor. The courses which precede it are concerned mainly with the development of concepts, the acquisition of rules for operation, and the perfection of skills, although

[1] W. H. Fagerstrom, Mathematical Facts and Processes Prerequisite to the Study of the Calculus, *Contributions to Education* 572 (New York: Bureau of Publications, Teachers College, Columbia University, 1933).

at some points there is an approach to real rigor in the treatment of certain theorems. On the other hand, the higher analytical courses are characterized by essentially rigorous and formal treatment of the subject matter. Thus, whether the student expects to "top off" his work in mathematics with calculus or to go on into the domain of higher mathematics, it is important that his study of calculus provide him, both as a matter of appreciation and as a matter of training, with some opportunity for really rigorous examination of certain topics.

Calculus offers numerous opportunities for such work in the analytical definitions associated with the concepts of limits, infinitesimals, continuity, differentials, derivatives, and the like, and in the proofs of certain theorems such as Rolle's Theorem, the Theorem of the Mean, the Fundamental Theorem of Definite Integrals, Taylor's Theorem, and Maclaurin's Theorem.

It is undoubtedly true that there are many students who will be able to do little with this kind of work, and it must not be assumed merely on this account that the course is worthless to such students. This, after all, is but one of the objectives of the course; it must not be forgotten that the use of a theorem and the proof of that same theorem are two entirely different matters and that things may often be extremely useful without being completely understood. On the other hand, those students who are able to appreciate the significance of this type of analysis and to follow its development will find a satisfying sense of security and finality in their work which must otherwise be lacking. Those who expect to go further in mathematics will find the training afforded in this sort of rigorous treatment of the foundations and theorems of calculus to be of inestimable value to them in their later work.

Exercises

1. Review and summarize the articles by (*a*) Farmer and (*b*) Kinney, indicated in the Bibliography for this chapter, and present a discussion of them to the class.
2. What have been the main arguments for the introduction of calculus in the high school?
3. Why, in your opinion, has calculus not been more extensively introduced in the high school?
4. Give a brief discussion of what may be done in high-school mathematics to prepare students to pursue more effectively their subsequent work in calculus in the college.
5. Give a precise statement of what you consider to be the objectives of the course in calculus in the junior college. Should these differ for the preengineering course and the general college course?
6. What do you consider the most serious indictment of the teaching of calculus as it is carried on in the colleges and engineering schools?
7. Discuss the role of formal work in calculus.

8. Give an intuitive explanation or illustration, and also a formal definition, of what is meant by saying that a variable approaches a constant as a limit. In teaching this topic, should the formal definition be given before or after the other explanation? Why?

9. Explain the difference between a derivative and a differential. What is an infinitesimal?

10. Given the function $y = f(x)$, explain why it is that $\Delta x = dx$ while in general $\Delta y \neq dy$.

11. Show why the study of calculus may be expected to enrich the student's mastery of algebra, trigonometry, and analytic geometry.

12. Give any mnemonic devices which you may be able to find which would help you to remember the fundamental formulas for derivatives.

13. Summarize the position indicated in this chapter as to the matter of precedence in proof of theorems or rules and the use of such theorems or rules. Illustrate.

14. Explain clearly the meaning of a second derivative, and explain its usefulness in the matter of determining maxima and minima and in the location and nature of inflection points.

15. Explain clearly the meanings of indefinite and definite integrals, giving illustrations.

16. What particular difficulties do students have in using an extensive table of integrals?

17. What is the test for the correctness of an indefinite integral?

18. Demonstrate the manner in which you would explain to a class the meaning of an element of integration, and the manner of setting up an expression of such an element. Use several applied problems by way of illustration.

19. Discuss the role of mathematical rigor in calculus.

20. Take several textbooks in calculus and after careful consideration decide which one, in your opinion, gives the most satisfactory explanation and treatment with reference to each of the following topics:

a. Limits, infinitesimals, and continuity

b. Increments and differentials

c. Derivatives and differentiation

d. Integration as antidifferentiation; indefinite integrals

e. Integration as summation; definite integrals

f. Multiple integrals

g. Infinite series, including tests for convergence and divergence

21. Give a review of the two articles by Whitman, listed in the Bibliography

22. Prepare a talk, as for a mathematics club, in which you will review the ideas set forth in Chap. 15 of Hooper's book "The River Mathematics" (see Bibliography).

23. Discuss the suggestions made in Humbert's article (see Bibliography), and give your opinion about them. Do you think the courses in calculus would be more effective if teachers would follow these suggestions as far as possible?

24. Pick out five things which seemed to you to present special difficulties when you were taking calculus. Subject these to careful analytical review, and try now to locate and identify the precise points which caused your difficulty. Then devise teaching procedures by which you would hope to make these things clearer to your own students.

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